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# BIOMETRIKA

## ON THE FREQUENCY DISTRIBUTION OF THE MEANS OF SAMPLES FROM A POPULATION HAVING ANY LAW OF FREQUENCY WITH FINITE MOMENTS, WITH SPECIAL REFERENCE TO PEARSON'S TYPE II.

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SUPPOSE a population to have $n_{x_1}$ individuals with a value $x_1$ of a character, $n_{x_2}$ with value $x_2, \ldots n_{x_s}$ with value $x_s, \ldots n_{x_w}$ with value $x_w$, and that we take samples at random containing $n$ individuals; the probability that the sum of the values of the character in a sample will be $X$ is the coefficient of $A^X$ in

$$\frac{(n_{x_1}A^{x_1} + n_{x_2}A^{x_2} + \ldots + n_{x_s}A^{x_s} + \ldots)^n}{N^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where $N$ is the total population. This may be written

$$(p_{x_1}A^{x_1} + p_{x_2}A^{x_2} + \ldots + p_{x_s}A^{x_s} + \ldots)^n \quad \ldots\ldots\ldots\ldots\ldots (1\,bis),$$

where $$p_{x_1} = \frac{n_{x_1}}{N},\ p_{x_2} = \frac{n_{x_2}}{N},\ \ldots,\ p_{x_s} = \frac{n_{x_s}}{N},\ \ldots,\ p_{x_w} = \frac{n_{x_w}}{N}.$$

This follows because there are $N^n$ ways of forming the sample, while the number which satisfies the condition that the total should be $X$ is clearly the coefficient of $A^X$ in the numerator of (1). It has been pointed out by Pearson that this theorem was originally due to Euler, but it is well known and has been used by H. E. Soper in his *Frequency Arrays**.

The theorem provides a ready method of obtaining the distribution of totals or means of samples from discrete populations; it has been shown by Pearson, for instance, that we may readily deduce from the theorem that the distribution of the means of samples of $r$ from the binomial $(p+q)^n$ plotted to unit $h$ is the binomial $(p+q)^{rn}$ plotted to unit $h/r$, and that a similar result is true for a Poisson series.

The purpose of the present paper is to adapt the process to obtaining the distributions of totals or means in samples from populations in which the variate has a continuous distribution.

The $r$th moment coefficient about the origin of the original population will be given by

$$p_{x_1}x_1^r + p_{x_2}x_2^r + p_{x_3}x_3^r + \ldots + p_{x_s}x_s^r + \ldots + p_{x_w}x_w^r,$$

which is equal to the coefficient of

$$\frac{\alpha^r}{r!} \text{ in } \sum_{s=1}^{s=w} p_{x_s}e^{\alpha x_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

* *Frequency Arrays*, p. 10, 1920. [Euler's merit lies in thinking of the problem long before the age was ripe for it. K. P.]

In H. E. Soper's elegant notation

$$\sum_{s=1}^{s=w} p_{x_s} A^{x_s}$$

would be termed the frequency array of the distribution in the original population, while

$$\sum_{s=1}^{s=w} p_{x_s} e^{ax_s}$$

would be its moment array.

The expression (1 *bis*) may be written after expansion

$$\sum_{t=1}^{t=v} P_{X_t} A^{X_t},$$

where $X_t$ is one of the possible values of the total (i.e. $n$ times the mean), $P_{X_t}$ is its corresponding probability and the summation is extended to all possible values $X_1, X_2, \ldots X_v$ of the totals. Thus

$$\left(\sum_{s=1}^{s=w} p_{x_s} A^{x_s}\right)^n = \left(\sum_{t=1}^{t=v} P_{X_t} A^{X_t}\right),$$

or on putting $A = e^a$

$$\left(\sum_{s=1}^{s=w} p_{x_s} e^{ax_s}\right)^n = \left(\sum_{t=1}^{t=v} P_{X_t} e^{aX_t}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

an expression in which we are really equating two different forms of the moment array of the distribution of totals.

Now if we suppose the distribution $\sum_{s=1}^{s=w} p_{x_s} A^{x_s}$ to pass into a continuous frequency distribution from $x = a$ to $x = b$ which may be denoted symbolically by

$$\int_a^b f(x)\, dx\, A^x,$$

the distribution of totals will also pass into a continuous distribution from $x = na$ to $x = nb$ which may be denoted by

$$\int_{na}^{nb} \psi(x)\, dX\, A^X,$$

and equation (3) will become

$$\left(\int_a^b f(x)\, e^{ax}\, dx\right)^n = \int_{na}^{nb} \psi(x)\, e^{ax}\, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Now if $f(x)$ is known (4) may be written

$$F(\alpha) = \int_{na}^{nb} \psi(x)\, e^{ax}\, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4\ bis),$$

and to obtain $\psi$ we have to solve the integral equation (4 *bis*).

Let $\alpha = i\beta$ and let $\phi(x)$ be a function which is equal to $\psi(x)$ when $na \leqslant x \leqslant nb$ and zero when $x$ is outside these limits, and let $F(i\beta) = \chi(\beta)$. Then

$$\chi(\beta) = \int_{-\infty}^{\infty} \phi(x)\, e^{i\beta x}\, dx,$$

and applying Fourier's Integral Theorem

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\beta)\, e^{-i\beta x} d\beta^*,$$

or

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x} F(i\beta)\, d\beta\dagger,$$

when

$$na \leqslant x \leqslant nb \ldots\ldots\ldots\

Now it may be shown* that

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2\lambda\alpha x)\, dx = e^{-\lambda\alpha^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx,$$

or
$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}n\beta^2} \cos(\beta x)\, d\beta = e^{-\frac{1}{2}\frac{x^2}{n}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}n\beta^2} d\beta$$
$$= \frac{\sqrt{2\pi}}{\sqrt{n}} e^{-\frac{1}{2}\frac{x^2}{n}},$$

or
$$\psi(x) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}\frac{x^2}{n}},$$

which gives the distribution of totals in samples of $n$.

Writing $n\bar{x} = x$,

$$\frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}\frac{x^2}{n}} dx \text{ becomes } \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2}n\bar{x}^2} n\, d\bar{x},$$

or
$$\frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2}\frac{\bar{x}^2}{\Sigma^2}} \text{ where } \Sigma = \frac{1}{\sqrt{n}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

This gives the frequency distribution of the means of samples taken at random from a normal population with unit standard deviation and is a result which is known otherwise to be correct.

(B) *Pearson's Type III.*

Now suppose $f(x) = \dfrac{e^{-x}x^{p-1}}{\Gamma(p)}$, $a = 0$, $b = \infty$.

Then
$$F(\alpha) = \left\{\int_0^{\infty} \frac{e^{-x}x^{p-1}}{\Gamma(p)} e^{\alpha x} dx\right\}^n$$
$$= \left\{\frac{1}{(1-\alpha)^p}\right\}^n = \frac{1}{(1-\alpha)^{np}}$$
$$= \frac{1}{(1-\alpha)^P} \text{ say, where } P = np.$$

Substituting this in (5)

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\beta x} d\beta}{(1-i\beta)^P}.$$

This integral is best evaluated by contour integration.

Let $Z = u + iv$ and let us integrate $\int \dfrac{e^{-ixz}\, dz}{(1-iz)^P}$ round a contour in the $z$ plane bounded by the real axis and a semi-circle $C$ of radius $R$ below it.

Then by Cauchy's Theorem :

$$\int_C \frac{e^{-ixz}\, dz}{(1-iz)^P} = 2\pi i \left(\text{the residue at the pole } z = \frac{1}{i}\right) = 2\pi i \left(\text{the coefficient of } \frac{1}{t} \text{ in the Laurent expansion of the integrand at } z = \frac{1}{i}\right).$$

* Whittaker and Watson, *Modern Analysis*, p. 114.

We obtain this by writing

$$z = \frac{1}{i} + t,$$

and expanding the integral we have

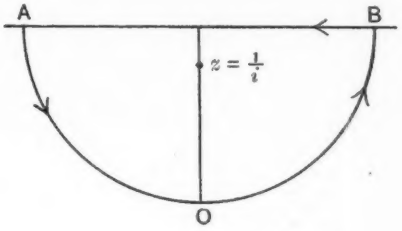
$$\frac{e^{-x(1+ti)}}{(-it)^P} = \frac{e^{-x}}{(-it)^P}\left\{1 - xti + \frac{(xti)^2}{2!} + \ldots\right\},$$

and the coefficient of

$$\frac{1}{t} = \frac{e^{-x}(xi)^{P-1}(-1)^{P-1}}{\Gamma(P)(-i)^P} = \frac{-e^{-x}x^{P-1}}{i\Gamma(P)},$$

therefore

$$\int_C \frac{e^{-ixz}dz}{(1-iz)^P} = \frac{-2\pi e^{-x}x^{P-1}}{\Gamma(P)}.$$




By making $R \to \infty$ it may be shown that provided $x > 0$*

$$\int_{AOB} \frac{e^{-ixz}dz}{(1-iz)^P} \to 0 \text{ as } R \to \infty,$$

and thus

$$+\int_{-\infty}^{\infty} \frac{e^{-ixu}du}{(1-iu)^P} = \frac{2\pi e^{-x}x^{P-1}}{\Gamma(P)}$$

and

$$\psi(x) = \frac{e^{-x}x^{P-1}}{\Gamma(P)}.$$

This gives the distribution of totals, for on putting $x = n\bar{x}$ we obtain

$$\frac{e^{-x}x^{P-1}}{\Gamma(P)}dx = \frac{ne^{-n\bar{x}}(n\bar{x})^{P-1}}{\Gamma(P)}d\bar{x}$$

$$= \frac{ne^{-n\bar{x}}(n\bar{x})^{np-1}}{\Gamma(np)}d\bar{x},$$

and the distribution of the means of samples from the curve $y = e^{-x}x^{p-1}/\Gamma(p)$ is given by†

$$y = \frac{ne^{-n\bar{x}}(n\bar{x})^{np-1}}{\Gamma(np)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

* When $x$ is negative it may be shown, by taking a semi-circular contour *above* the real axis, that $\int_{-\infty}^{\infty} \frac{e^{-ixu}}{(1-iu)^P}du = 0$ as it should do.

† A more direct method of reaching this result has already been given in a paper by A. E. R. Church, *Biometrika*, Vol. XVIII. pp. 335—338.

(C) *Pearson's Type II.*

Let us put $f(x) = x^{p-1}(1-x)^{p-1}$, where $p > 0$.

Then $$F(\alpha) = \left\{\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x} dx\right\}^n.$$

Put $x = \sin^2\theta$, then

$$\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x}\, dx = \int_0^{\pi/2} \sin^{2p-2}\theta \cos^{2p-2}\theta e^{\alpha \sin^2\theta} (2\sin\theta\cos\theta\, d\theta)$$

$$= 2\int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2p-1}\theta e^{\alpha\sin^2\theta}\, d\theta.$$

On putting $2\theta = \phi$ we have

$$\int_0^1 x^{p-1}(1-x)^{p-1} e^{\alpha x}\, dx = \frac{1}{2^{2p-1}}\int_0^{\pi} \sin^{2p-1}\phi e^{\frac{\alpha}{2}(1-\cos\phi)}\, d\phi$$

$$= \frac{e^{\frac{\alpha}{2}}}{2^{2p-1}}\int_0^{\pi} \sin^{2p-1}\phi e^{-\frac{\alpha}{2}\cos\phi}\, d\phi.$$

Now if $I_n(z)$ is Bessel's function of the $n$th order and imaginary argument it is known that*

$$I_n(z) = \frac{z^n}{2^n \Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}\int_0^{\pi} e^{-z\cos\phi}\sin^{2n}\phi\, d\phi,$$

provided $n + \frac{1}{2} > 0$.

Thus $$\int_0^{\pi} e^{-\frac{\alpha}{2}\cos\phi}\sin^{2p-1}\phi\, d\phi = \frac{2^{p-\frac{1}{2}}\Gamma(p)\Gamma(\frac{1}{2})}{\left(\frac{\alpha}{2}\right)^{p-\frac{1}{2}}} I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right),$$

provided $p > 0$; and

$$\int_0^1 f(x) e^{\alpha x}\, dx = e^{\frac{\alpha}{2}}\Gamma(p)\Gamma(\tfrac{1}{2})\alpha^{-p+\frac{1}{2}} I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right) \quad \text{.....(8),}$$

and $$F(\alpha) = \frac{\{\Gamma(p)\}^n\{\Gamma(\frac{1}{2})\}^n}{\alpha^{n(p-\frac{1}{2})}} e^{\frac{n\alpha}{2}}\left\{I_{p-\frac{1}{2}}\left(\frac{\alpha}{2}\right)\right\}^n \quad \text{.....(9).}$$

Substituting in (5)†

$$\psi(x) = \frac{\{\Gamma(p)\}^n\{\Gamma(\frac{1}{2})\}^n}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x} e^{\frac{ni\beta}{2}} (i\beta)^{-n(p-\frac{1}{2})}\left[I_{p-\frac{1}{2}}\left\{\frac{i\beta}{2}\right\}\right]^n d\beta,$$

when $0 \leqslant x \leqslant n$.

* Whittaker and Watson, *Modern Analysis*, p. 360.

† In order that this solution of the integral equation may be valid, it is necessary to show that

$$\int_{-\infty}^{\infty} \frac{e^{-i\beta\left(x-\frac{n}{2}\right)}}{\beta^{n(p-\frac{1}{2})}}\left[J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right]^n d\beta = 0,$$

if $x > n$ or negative.

Writing $$x - \frac{n}{2} = y,$$

it must be shown that

$$\int_{-\infty}^{\infty}\phi(\beta)\, d\beta = \int_{-\infty}^{\infty}\frac{e^{-i\beta y}}{\beta^{n(p-\frac{1}{2})}}\left[J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right]^n d\beta = 0 \text{ if } y > \frac{n}{2} \text{ or } < -\frac{n}{2}.$$

Now $$I_{p-\frac{1}{2}}\left(\frac{i\beta}{2}\right) = i^{p-\frac{1}{2}} J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right).$$

Thus $$\psi(x) = \frac{\{\Gamma(p)\}^n \{\Gamma(\frac{1}{2})\}^n}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\beta\left(x-\frac{n}{2}\right)}}{\beta^{n(p-\frac{1}{2})}} \left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n d\beta \ldots\ldots\ldots(10).$$

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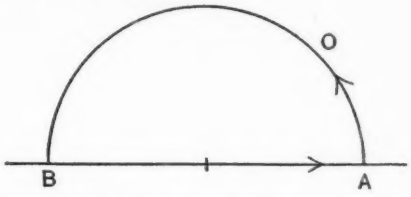
I am indebted to Mr E. C. Titchmarsh for outlining the following proof that this is true.

Taking $\beta$ as a complex variable let us integrate the above integral round a semi-circular contour with centre at the origin and lying above the real axis.




The function has no singularities within the contour, thus

$$\int_{-\infty}^{\infty} \phi(\beta)\, d\beta = -\lim_{|\beta|\to\infty} \int_{AOB} \phi(\beta)\, d\beta.$$

Let $$\beta = \lambda + i\mu.$$

Then $$|e^{-i\beta y}| = |e^{-iy(\lambda+i\mu)}| = e^{\mu y}.$$

Now $$\left|\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n\right| \sim \left|\frac{\cos\left(\frac{\beta}{2}\right)}{\sqrt{\frac{\beta}{2}}}\right|^n$$

$$\sim \frac{e^{\frac{\mu n}{2}}}{|\beta|^{\frac{n}{2}}}.$$

Thus $$|\phi(\beta)| \sim \frac{e^{\mu\left(y+\frac{n}{2}\right)}}{|\beta|^{np}}.$$

Thus if $y < -\frac{n}{2}$, remembering $p = 0$,

$$|\phi(\beta)| < \frac{Ae^{-kr\sin\theta}}{r},$$

for sufficiently large $r$ where $\lambda = r\cos\theta$, $\mu = r\sin\theta$ and $A$ is a constant, and thus

$$\int_{AOB} \phi(\beta)\, d\beta \to 0 \text{ as } r \to \infty,$$

therefore $$\int_{-\infty}^{\infty} \phi(\beta)\, d\beta = 0.$$

If we take a similar contour below the real axis

$$|e^{-i\beta y}| = e^{\mu y}$$

as before, but

$$\left|\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n\right| \sim \frac{e^{-\frac{\mu n}{2}}}{\left|\sqrt{\frac{\beta}{2}}\right|^n},$$

Now $$J_n(z)=\frac{z^n}{2^n\,\Gamma(n+1)}\left\{1-\frac{(z/2)^2}{1!\,(n+1)}+\frac{(z/2)^4}{2!\,(n+1)(n+2)}-\ldots\right\}.$$

Thus $$J_{p-\frac{1}{2}}(z)=\frac{z^{p-\frac{1}{2}}}{2^{p-\frac{1}{2}}\,\Gamma(p+\frac{1}{2})}\left\{1-\frac{(z/2)^2}{1!\,(p+\frac{1}{2})}+\frac{(z/2)^4}{2!\,(p+\frac{1}{2})(p+\frac{3}{2})}-\ldots\right\},$$

therefore $$\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}=\text{Const.}\times(\text{Series in even powers of }\beta).$$

Accordingly $$\int_{-\infty}^{\infty}\sin\left(x-\frac{n}{2}\right)\beta\,\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\,d\beta=0,$$

since the integrand is an odd function of $\beta$.

We now have

$$\psi(x)=\tfrac{1}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\int_{-\infty}^{\infty}\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\cos\left[\left(x-\frac{n}{2}\right)\beta\right]d\beta\ldots\ldots(11).$$

On transferring the origin to the mean of the distribution of totals, i.e. putting $x=\frac{n}{2}+x'$, $\psi(x)=w(x')$, we obtain the simpler form

$$w(x')=\tfrac{1}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\int_{-\infty}^{\infty}\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\cos(x'\beta)\,d\beta\quad\ldots\ldots(12).$$

In particular when $p=1$ in virtue of the relation

$$J_{\frac{1}{2}}(z)=z^{\frac{1}{2}}\sqrt{\frac{2}{\pi}}\frac{\sin z}{z},$$

we have $$w(x')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n\cos(x'\beta)\,d\beta\quad\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

giving the distribution of totals when in the original population all values between $x=0$ and $x=1$ are equally likely.

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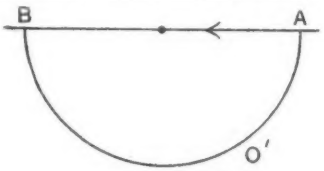



and $$|\phi(\beta)|\sim\frac{e^{\mu\left(y-\frac{n}{2}\right)}}{|\beta|^{np}}.$$

Thus if $$y>\frac{n}{2}\quad|\phi(\beta)|<\frac{Ae^{kr\sin\theta}}{r},$$

for sufficiently large $r$, and remembering that $\sin\theta$ is negative

$$\int_{ABO'}\phi(\beta)\,d\beta\to 0\text{ as }r\to\infty$$

and $$\int_{-\infty}^{\infty}\phi(\beta)\,d\beta=0$$

as before.

The relation (13) might have been proved *ab initio* as follows:

If $p=1$, $f(x)=1$ and the moment array of the original population about the mean is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\alpha x}\,dx = \frac{\sinh\frac{\alpha}{2}}{\frac{\alpha}{2}}.$$

Thus
$$F(\alpha) = \left(\frac{\sinh\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^n,$$

and using the corresponding relation to (5) with the mean as origin we obtain

$$w(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x}\left(\frac{i\sin\frac{\beta}{2}}{\frac{i\beta}{2}}\right)^n d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x'}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n d\beta,$$

and remembering $\frac{\sin\theta}{\theta}$ is an even function of $\theta$ we have

$$\int_{-\infty}^{\infty} \sin\beta x'\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n d\beta = 0,$$

therefore
$$w(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n \cos(x'\beta)\,d\beta \quad \text{.....(13 \textit{bis})}.$$

The agreement of (13) and (13 *bis*) gives a check on the previous work.

The distribution of means is now easily obtained, writing $x' = n\bar{x}$ we have

$$w(x')\,dx' = \frac{n}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\,d\bar{x}\int_{-\infty}^{\infty}\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\cos(n\bar{x}\beta)\,d\beta,$$

so that the distribution of means is given by

$$y = \frac{n}{2}(\sqrt{\pi})^{n-2}\{\Gamma(p)\}^n\int_{-\infty}^{\infty}\frac{\left\{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)\right\}^n}{\beta^{n(p-\frac{1}{2})}}\cos(n\bar{x}\beta)\,d\beta \quad \text{.....(14)},$$

and in particular when $p=1$

$$y = \frac{n}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right)^n \cos(n\bar{x}\beta)\,d\beta \quad \text{.....(15)}.$$

We will now consider this special case in some detail.

The integral (13) admits of exact determination. On writing $\frac{\beta}{2} = \theta$ (13) becomes

$$w(x') = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin^n \theta}{\theta^n} \cos(2x'\theta)\, d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

There are two cases to be considered: (i) when $n$ is even, (ii) when $n$ is odd.

(i) Let $n$ be even $= 2r$ say.

Now by writing $\cos\theta + i\sin\theta = t$, we have the identity

$$(2i\sin\theta)^{2r} = \left(t - \frac{1}{t}\right)^{2r},$$

from which it is easily shown that

$$\sin^{2r}\theta = \frac{(-1)^r}{2^{2r-1}}\{\cos 2r\theta - {}^{2r}C_1 \cos(2r-2)\theta + \ldots + (-1)^s\, {}^{2r}C_s \cos 2(r-s)\theta + \ldots + (-1)^{r-1}\, {}^{2r}C_{r-1}\cos 2\theta + \tfrac{1}{2}(r)^r\, {}^{2r}C_r\}$$

Now $\quad \cos 2s\theta \cos 2x\theta = \frac{1}{2}\{\cos 2(s+x)\theta + \cos 2(s-x)\theta\}.$

Thus

$$\sin^{2r}\theta \cos 2x\theta = \frac{(-1)^r}{2^{2r-1}} \sum_0^{r-1} \tfrac{1}{2}\, {}^{2r}C_s [\cos 2(r-s+x)\theta + \cos 2(r-s-x)\theta] + \frac{1}{2^r}\, {}^{2r}C_r \cos 2x\theta,$$

and in order to integrate (16) we shall require

$$\int_{-\infty}^{\infty} \frac{\cos 2k\theta}{\theta^n}\, d\theta = \left[\frac{-\cos 2k\theta}{(n-1)\,\theta^{n-1}}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{2k\sin 2k\theta}{(n-1)\,\theta^{n-1}}\, d\theta$$

$$= \frac{-2k}{(n-1)}\int_{-\infty}^{\infty} \frac{\sin 2k\theta}{\theta^{n-1}}\, d\theta = \frac{-2k}{(n-1)}\left\{\left[\frac{-\sin 2k\theta}{(n-2)\,\theta^{n-2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{2k\cos 2k\theta}{(n-2)\,\theta^{n-2}}\, d\theta\right\}$$

$$= \frac{-2k}{(n-1)}\int_{-\infty}^{\infty} \frac{2k\cos 2k\theta}{(n-2)\,\theta^{n-2}}\, d\theta$$

$$= \frac{-(2k)^2}{(n-1)(n-2)}\int_{-\infty}^{\infty} \frac{\cos 2k\theta}{\theta^{n-2}}\, d\theta.$$

Thus $\quad \int_{-\infty}^{\infty} \frac{\cos 2k\theta\, d\theta}{\theta^{2r}} = \frac{(-1)^{r-1}(2k)^{2r-2}}{(2r-1)(2r-2)\ldots 2}\int_{-\infty}^{\infty} \frac{\cos 2k\theta}{\theta^2}\, d\theta,$

and $\quad \int_{-\infty}^{\infty} \frac{\cos 2k\theta\, d\theta}{\theta^2} = \left[-\frac{\cos 2k\theta}{\theta} - \int\frac{2k\sin 2k\theta}{\theta}\, d\theta\right]_{-\infty}^{\infty}$

$$= -\int_{-\infty}^{\infty} \frac{2k\sin 2k\theta}{\theta}\, d\theta.$$

Now $$\int_{-\infty}^{\infty} \frac{\sin 2k\theta}{\theta}\, d\theta = \pi \text{ if } 2k > 0, \; -\pi \text{ if } 2k < 0,$$

therefore $$\int_{-\infty}^{\infty} \frac{\cos 2k\theta\, d\theta}{\theta^{2r}} = \frac{(-1)^r (2k)^{2r-1}}{(2r-1)!}\pi \text{ if } 2k > 0,$$

and $$= \frac{(-1)^{r-1} (2k)^{2r-1}}{(2r-1)!}\pi \text{ if } 2k < 0.$$

Thus $$\int_{-\infty}^{\infty} \sin^{2r}\theta \cos 2x\theta \frac{d\theta}{\theta^{2r}}$$

$$= \frac{(-1)^r}{2^{2r}} \int_{-\infty}^{\infty} \left[ \sum_0^{r-1} (-1)^s {}^{2r}C_s \{\cos 2(r-s+x)\theta + \cos 2(r-s-x)\theta\} + (-1)^r {}^{2r}C_r \cos 2x\theta \right] \frac{d\theta}{\theta^{2r}}$$

$$= \frac{1}{2^{2r}} \left[ \sum_0^{r-1} (-1)^s {}^{2r}C_s \left\{ \pm \frac{[2(r-s+x)]^{2r-1}}{(2r-1)!} \pm \frac{[2(r-s-x)]^{2r-1}}{(2r-1)!} \right\} + (-1)^r {}^{2r}C_r \left[ \pm \frac{(2x)^{2r-1}}{(2r-1)!} \right] \right] \pi$$

$$= \frac{\pi/2}{2r-1!} \left[ \sum_0^{r-1} (-1)^s {}^{2r}C_s \{ \pm (r-s+x)^{2r-1} \pm (r-s-x)^{2r-1} \} + (-1)^r {}^{2r}C_r [\pm x^{2r-1}] \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

where the positive sign is to be taken in front of a bracket if the expression within the bracket is itself positive and the negative sign in the contrary case.

Thus if $n$ is even

$$\int_{-\infty}^{\infty} \frac{\sin^n \theta}{\theta^n} \cos 2x'\theta\, d\theta = \frac{\pi/2}{(n-1)!} \left[ \sum_0^{\frac{n}{2}-1} (-1)^s {}^nC_s \left\{ \pm \left(\frac{n}{2} - s + x'\right)^{n-1} \pm \left(\frac{n}{2} - s - x'\right)^{n-1} \right\} + (-1)^{n/2} {}^nC_{n/2} [\pm x']^{n-1} \right],$$

with the above rule with regard to signs.

(ii) Let $n$ be odd $= (2r-1)$ say.

Then it is easily shown that

$$\sin^{2r-1}\theta = \frac{(-1)^{r-1}}{2^{2r-2}} \{\sin(2r-1)\theta - {}^{2r-1}C_1 \sin(2r-3)\theta + \ldots + (-1)^s {}^{2r-1}C_s \sin(2r-2s-1)\theta + \ldots + (-1)^{r-1} {}^{2r-1}C_{r-1} \sin\theta\},$$

and

$$\sin(2r-2s-1)\theta \cos 2x\theta = \tfrac{1}{2}\{\sin[2(r-s+x)-1]\theta + \sin[2(r-s-x)-1]\theta\},$$

and in order to perform the integration we shall require

$$\int_{-\infty}^{\infty} \frac{\sin(2k-1)\theta\, d\theta}{\theta^{2r-1}}.$$

By the same method as employed above it is now easily shown that

$$\int_{-\infty}^{\infty} \frac{\sin(2k-1)\theta\, d\theta}{\theta^{2r-1}} = \frac{(-1)^{r-1}(2k-1)^{2r-2}}{(2r-2)!}\pi \text{ if } 2k-1>0,$$

and $$= \frac{(-1)^{r}(2k-1)^{2r-2}}{(2r-2)!}\pi \text{ if } 2k-1<0;$$

therefore

$$\int_{-\infty}^{\infty} \sin^{2r-1}\theta \cos 2x\theta \frac{d\theta}{\theta^{2r-1}}$$

$$= \frac{(-1)^{r-1}}{2^{2r-1}} \int_{-\infty}^{\infty} \left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s \{\sin[2(r-s+x-1)-1]\theta + \sin[2(r-s-x)-1]\theta\}\right] \frac{d\theta}{\theta^{2r-1}}$$

$$= \frac{(-1)^{r-1}}{2^{2r-1}}\pi \left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s (-1)^{r-1} \left\{\pm \frac{[2(r-s+x)-1]^{2r-2}}{(2r-2)!} \pm \frac{[2(r-s-x)-1]^{2r-2}}{(2r-2)!}\right\}\right]$$

$$= \frac{\pi/2}{2r-2!}\left[\sum_{s=0}^{r-1} (-1)^{s\,2r-1}C_s \{\pm[r-s+x-\tfrac{1}{2}]^{2r-2} \pm [r-s-x-\tfrac{1}{2}]^{2r-2}\}\right] \quad ...(18),$$

where the positive sign is to be taken in front of a bracket if the expression within the bracket is itself positive and the negative sign in the contrary case.

Thus if $n$ is odd

$$\int_{-\infty}^{\infty} \left(\frac{\sin\theta}{\theta}\right)^n \cos 2x'\theta\, d\theta = \frac{\pi/2}{(n-1)!}\left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm\left(\frac{n}{2}-s+x'\right)^{n-1} \pm\left(\frac{n}{2}-s-x'\right)^{n-1}\right\}\right]$$

with the above rule with regard to signs.

If the sampled population be one in which all values of a variate between 0 and 1 are equally likely the equation (referred to the mean) of the distribution of the means of random samples of $n$ will be given by

$$y = \frac{n}{n-1!}\left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm\left(\frac{n}{2}-s+n\bar{x}\right)^{n-1} \pm\left(\frac{n}{2}-s-n\bar{x}\right)^{n-1}\right\}\right] + (-1)^{n/2\,n}C_{n/2}\{\pm(n\bar{x})\}^{n-1} \quad .....(19),$$

if $n$ is even and

$$y = \frac{n}{n-1!}\left[\sum_{s=0}^{\frac{n-1}{2}} (-1)^{s\,n}C_s \left\{\pm\left(\frac{n}{2}-s+n\bar{x}\right)^{n-1} \pm\left(\frac{n}{2}-s-n\bar{x}\right)^{n-1}\right\}\right] \quad ...(19\ bis),$$

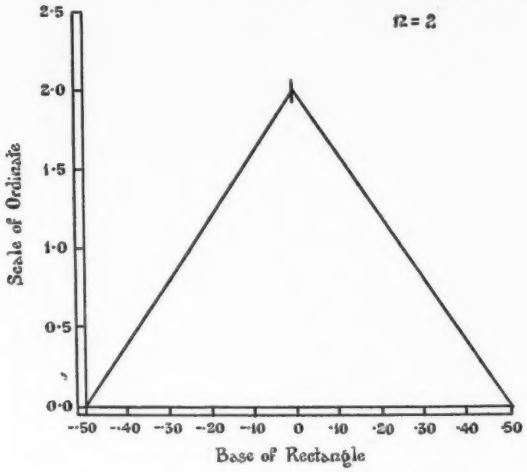
if $n$ is odd, and the ambiguous sign is to be determined as explained above. The range will be from $\bar{x}=-\frac{1}{2}$ to $\bar{x}=+\frac{1}{2}$.

On putting $n=2, 3, 4$ in these formulae, paying attention to the above rule of signs, we obtain the following results for the frequency distribution of means in samples of 2, 3, 4 from a population in which all values of $x$ between $-\frac{1}{2}$ and $+\frac{1}{2}$ are equally likely.

$n = 2$. $y = 2\{1 - 2|x|\}$,

or the frequency distribution is an isosceles triangle.

FIG. I

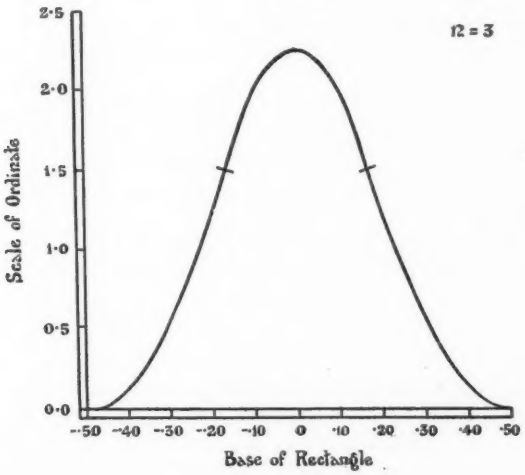



$n = 3$. $y = \frac{9}{4}(1 - 12x^2)$ if $-\frac{1}{6} < x < \frac{1}{6}$,

$y = \frac{27}{8}(1 - 2|x|)^2$ if $x > \frac{1}{6}$ or $< -\frac{1}{6}$,

or the frequency distribution is made up of three parabolas and has something of the appearance of a normal curve.

FIG. II

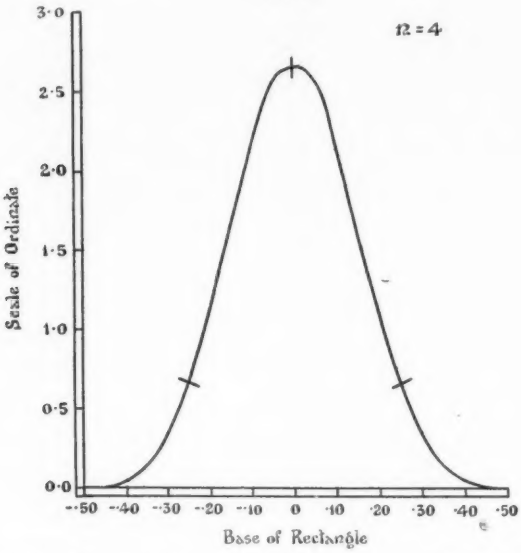


$n = 4$. $\quad y = \frac{8}{3}\,[1 - 24x^2 + 48\,|x|^3]$ if $-\frac{1}{4} < x < \frac{1}{4}$,

$y = \frac{16}{3}\,[1 - 2\,|x|]^3$ if $x > \frac{1}{4}$ or $< -\frac{1}{4}$,

and the frequency distribution is made up of four cubic curves with a point of discontinuity at the vertex and two points where differential coefficients are discontinuous at $x = \frac{1}{4}$ and $x = -\frac{1}{4}$.

FIG. III




It is clear that when $n = 5$, the distribution will be made up of five curves of the 4th degree joining at the points $x = \pm\frac{1}{10}, \pm\frac{3}{10}$ and in general that

(i) The distribution of means is symmetrical.

(ii) If $n$ be even there is a point of abruptness at the vertex, but if $n$ be odd there is not.

(iii) There are $n - 1$ points of abruptness or points where differential coefficients become discontinuous.

(iv) These points of abruptness gradually become less marked as $n$ increases and the appearance of the curves approaches rapidly to the normal. In fact when $n$ is 10, $\beta_2 = 2{\cdot}88$*.

* This may be shown as follows:

The moment array of (19) about the mean is, as we have already seen, given by

$$F(\alpha) = \left(\frac{\sinh\frac{\alpha}{2}}{\frac{\alpha}{2}}\right)^n$$

$$= \left\{\frac{\frac{\alpha}{2} + \frac{1}{3!}\left(\frac{\alpha}{2}\right)^3 + \frac{1}{5!}\left(\frac{\alpha}{2}\right)^5 + \ldots}{\frac{\alpha}{2}}\right\}^n$$

$$= 1 + \frac{n}{12}\frac{\alpha^2}{2!} + \frac{\alpha^4}{4!}\left(\frac{n}{80} - \frac{n}{48} + \frac{n^2}{48}\right),$$

We now return to the general case given by (14). It is known that

$$J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos z,$$

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin z}{z^{\frac{1}{2}}},$$

and when $p$ is a positive integer

$$J_{p-\frac{1}{2}}(z) = \frac{(-1)^{p-1}(2z)^{p-\frac{1}{2}}}{\pi^{\frac{1}{2}}} \frac{d^{p-1}}{d(z^2)^{p-1}} \left(\frac{\sin z}{z}\right)^*.$$

Thus

$$\frac{J_{-\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{-\frac{1}{2}}} = \frac{2}{\sqrt{\pi}} \cos\frac{\beta}{2},$$

$$\frac{J_{\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{\frac{1}{2}}} = \frac{2}{\sqrt{\pi}} \frac{\sin\frac{\beta}{2}}{\beta},$$

$$\frac{J_{p-\frac{1}{2}}\left(\frac{\beta}{2}\right)}{\beta^{p-\frac{1}{2}}} = \frac{(-1)^{p-1}}{\sqrt{\pi}} \frac{2^{2p-2} d^{p-1}}{d(\beta^2)^{p-1}} \left(\frac{\sin\frac{\beta}{2}}{\frac{\beta}{2}}\right).$$

It is thus possible by substitution of this expression in (14) and simplifying to reduce the integrand to the sum of terms of the type $\frac{\sin k\beta}{\beta^s}$ or $\frac{\cos k\beta}{\beta^s}$ which are each integrable. Thus (14) is integrable when $p$ is a positive integer and for low values of $n$ and $p$ (say up to 4 or 5) it should be quite possible to evaluate the integrals†. For higher values of $n$ the number of terms in the simplified expression would render the process more difficult, but since when $p$ is a positive integer the distribution of means will approach more rapidly to the normal than in the case of the rectangle treated above, we should not in practice require to take $n$ greater than at most 10 and $p$ than 3 or 4. This should be sufficient to give a good idea in practice of the nature of the distribution of means of small samples taken at random from the Type II curve.

therefore $\mu_2 = \frac{n}{12}$, $\mu_4 = \frac{1}{240}(5n^2 - 2n)$.

Whence $\beta_1 = 0$, $\beta_2 = 3 - \frac{6}{5n}$,

and $\beta_2 \to 3$ as $n \to \infty$.

and in particular when $n = 10$ $\beta_2 = 2{\cdot}88$,

when $n = 50$ $\beta_2 = 2{\cdot}976$.

* Whittaker and Watson, *Modern Analysis*, 2nd Edition, p. 358.

† It is hoped that at a later date this may be done.

# THE DISTRIBUTION OF MEANS FOR SAMPLES OF SIZE $N$ DRAWN FROM A POPULATION IN WHICH THE VARIATE TAKES VALUES BETWEEN 0 AND 1, ALL SUCH VALUES BEING EQUALLY PROBABLE.

By PHILIP HALL, B.A.

A particular sample in which the variate $x$ takes the values $x_1, x_2, \ldots, x_N$ may be represented by the point $P = (x_1, x_2, \ldots, x_N)$ of $N$-dimensional space. The representative points of a large number of samples will tend to be evenly distributed throughout the interior of the unit hypercube

$$0 \leqslant x_i \leqslant 1, \quad (i = 1, 2, \ldots, N) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let $O$ be the origin and $I$ the point $(1, 1, \ldots, 1)$. Then if $M$ is the foot of the perpendicular from $P$ on to $OI$, we have

$$\text{Mean of sample} = m = \frac{OM}{OI} \quad \ldots\ldots\ldots (2),$$

and

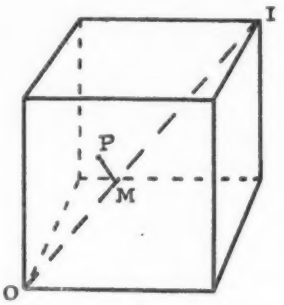
$$\text{Standard deviation of sample} = \sigma = \frac{MP}{OI}.$$

We shall write $p = Nm$ ..... (3).

Then the points whose mean is equal to $m$ all lie in the region cut off by (1) on the hyperplane

$$\sum_{i=1}^{N} x_i = p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and conversely. Hence in order to find the frequency distribution of $m$ it is only necessary to calculate the $N-1$-dimensional content of this region.




With this object, we consider the "quadrants"

$$\begin{matrix} x_i \geqslant r_i, \\ r_i = 0 \text{ or } 1, \end{matrix} \quad (i = 1, 2, \ldots, N) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

whose corners are the corners of (1). These we divide into $N+1$ sets according to the value of

$$r = \sum_{i=1}^{N} r_i \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

so that $r = 0, 1, 2, \ldots, N$. A quadrant of the $i$th set will be called a $Q_i$. Thus the number of $Q_i$'s is $\binom{N}{i}$.

Let $S$ be any point of $Q_0$, i.e. any point whose coordinates are all $\geqslant 0$, and let just $s$ of its coordinates be $\geqslant 1$. Then $S$ will belong to just $\binom{s}{1}$ $Q_1$'s, $\binom{s}{2}$ $Q_2$'s, ... and $\binom{s}{s} = 1 Q_s$. Now, if $s > 0$, we have

$$\sum_{r=0}^{s} (-1)^r \binom{s}{r} = 0 \quad \text{.....(7).}$$

Hence if whenever a point of (4) belongs to a $Q_r$ we give it a "density" $(-1)^r$ and then sum over all $Q$, the resultant density will be equal to 1 or 0 according as the point in question belongs to (1) or not. Let the segment of (4) lying in $Q_0$ have the content $V_N(p)$. Then the segment of (4) lying in (5) will have the content $V_N(p-r)$, (which is 0 when $r \geqslant p$). And the segment of (4) lying in (1) will have the content

$$\sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} V_N(p-r) \quad \text{.....(8),}$$

where $\kappa = [p]$ is the greatest integer less than $p$.

It remains to find $V_N(p)$. Let $V_{N-1}(p)$ be the content of the projection of $V_N(p)$ perpendicular to one of the axes, so that

$$V_N(p) = \sqrt{N} \times \overline{V}_{N-1}(p).$$

Now $\overline{V}_N(p)$ is the content of the $N$-dimensional region bounded by (4) and the coordinate hyperplanes, a region whose base is therefore of content $V_N(p)$. The perpendicular from $O$ on to this base is equal to $p/\sqrt{N}$. Hence

$$\overline{V}_N(p) = \frac{1}{N} \times \frac{p}{\sqrt{N}} \times V_N(p) \quad \text{.....(9).}$$

Since $V_2(p) = \sqrt{2}\,p$, we get

$$V_N(p) = \frac{\sqrt{N}}{(N-1)!} p^{N-1} \quad \text{.....(10).}$$

Substituting in (8), we find for the content of the region common to (1) and (4) the value

$$f(p) = \frac{\sqrt{N}}{(N-1)!} \sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} (p-r)^{N-1} \quad \text{.....(11),}$$

for values of $p$ between $\kappa$ and $\kappa+1$. To find the actual frequency distribution of $m$, we note that $\int_0^N f(p)\, d\left(\frac{p}{\sqrt{N}}\right) =$ the volume of the hypercube $= 1$ and hence $\int_0^1 f(Nm)\, dm = \frac{1}{\sqrt{N}}$. The distribution $F(m)$ of $m$ is, therefore, given by

$$\left.\begin{aligned} F(m) &= \frac{N^N}{(N-1)!} \sum_{r=0}^{\kappa} (-1)^r \binom{N}{r} \left(m - \frac{r}{N}\right)^{N-1} \\ \frac{\kappa}{N} &\leqslant m \leqslant \frac{\kappa+1}{N} \end{aligned}\right\} \quad \text{.....(12).}$$

Naturally $y = F(m)$ is symmetrical about $m = \frac{1}{2}$. It consists of $N$ arcs of degree $N-1$ having $N-1$-point contact at their joins, viz. at the points

$$\frac{\kappa}{N} (\kappa = 1, 2, \ldots, N-1).$$

*The Moments.*

These are defined by

$$\bar{M}_\alpha = \int_0^1 (m - \tfrac{1}{2})^\alpha F(m)\, dm \quad \ldots\ldots(13)$$

Owing to the symmetry of $F$, we have

$$\bar{M}_{2\alpha+1} = 0, \quad (\alpha = 0, 1, \ldots) \quad \ldots\ldots(14),$$

while
$$\bar{M}_{2\alpha} = \frac{2N^N}{(N-1)!} \sum_{r=0}^{\left[\frac{N}{2}\right]} (-1)^r \binom{N}{r} \int_{\frac{r}{N}}^{\frac{1}{2}} (m - \tfrac{1}{2})^{2\alpha} \left(m - \frac{r}{N}\right)^{N-1} dm$$

$$= \frac{2N^N}{(N-1)!} \sum_{r=0}^{\left[\frac{N}{2}\right]} (-1)^r \binom{N}{r} \left(\frac{1}{2} - \frac{r}{N}\right)^{N+2\alpha} B(2\alpha+1, N),$$

or using $\Delta^N$ to denote the $n$th difference,

$$\bar{M}_{2\alpha} = \frac{B(2\alpha+1, N)}{(N-1)!\, N^{2\alpha}} \sum_{r=0}^{N} (-1)^r \binom{N}{r} \left(\frac{N}{2} - r\right)^{N+2\alpha} \quad \ldots\ldots(15),$$

$$\bar{M}_{2\alpha} = \frac{(2\alpha)!}{(N+2\alpha)!\, N^{2\alpha}} \Delta^N_{x=-\frac{N}{2}} (x^{N+2\alpha}) \quad \ldots\ldots(15)',$$

or, in terms of the generalised polynomials of Bernoulli* defined by

$$B_\nu^{(-N)}(x) = \frac{\nu!}{(N+\nu)!} \Delta^N x^{N+\nu} \quad \ldots\ldots(16),$$

$$\bar{M}_{2\alpha} = \frac{1}{N^{2\alpha}} B_{2\alpha}^{(-N)}\left(-\frac{N}{2}\right) \quad \ldots\ldots(17).$$

These moments can, however, be obtained in a different form and by an entirely different method.

We consider an *arbitrary* frequency distribution $f(x)$ of a variate $x$ and write

$$\mu_\alpha = \int_{-\infty}^{+\infty} x^\alpha f(x)\, dx \quad \ldots\ldots(18).$$

We may suppose $\mu_0 = 1$ and $\mu_1 = 0$. Then the distribution of means $m = \frac{1}{N} \sum_{i=1}^{N} x_i$ for samples of size $N$ will be given by

$$F(m) = A \int f_1 f_2 \ldots f_N d\tau \quad \ldots\ldots(19),$$

where $f_i = f(x_i)$ and $d\tau$ is the $N-1$-dimensional volume-element in the hyperplane (4), the integral being taken completely over (4). Let us denote by $M_\alpha$ the moments of this distribution. Then

$$M_0 = \int_{-\infty}^{+\infty} F(m)\, dm = 1,$$

if $A = \sqrt{N}$, which we accordingly suppose, and

$$M_\alpha = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} m^\alpha f_1 f_2 \ldots f_N dx_1 \ldots dx_N,$$

i.e.
$$M_\alpha = \frac{1}{N^\alpha} \Sigma \frac{\alpha!}{s_1!\, s_2! \ldots s_N!} \mu_{s_1} \mu_{s_2} \ldots \mu_{s_N} \quad \ldots\ldots(20),$$

* Not to be confused with the $B$ of the complete Beta-function in formula (15).

the summation being taken over all integer solutions of

$$\left.\begin{aligned} \sum_{i=1}^{N} s_i &= \alpha \\ s_i &\geqslant 0, \quad (i = 1, 2, \ldots, N) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21).$$

Since $\mu_1 = 0$, we may also add the condition $s_i \neq 0$. Now in our case

$$\left.\begin{aligned} \mu_{2s-1} &= 0 \quad (s = 1, 2, \ldots) \\ \mu_{2s} &= \int_0^1 (m - \tfrac{1}{2})^{2s} dm = \frac{1}{2s+1} (\tfrac{1}{2})^{2s} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Hence, we get

$$\bar{M}_{2\alpha} = \frac{1}{(2N)^{2\alpha}} \Sigma \frac{(2\alpha)!}{(2s_1+1)!\,(2s_2+1)!\ldots(2s_N+1)!} \quad \ldots\ldots\ldots\ldots (23),$$

taken as before over all integer solutions of (21)*.

We return now to the general formula (20) true for any frequency distribution. To express $M_\alpha$ as a polynomial in the $\mu$, let us take any partition of $\alpha$ into $N$ non-negative summands, and let there be $a_0$ 0's, $a_1$ 1's, $a_2$ 2's and so on, so that

$$\Sigma a_i = N, \quad \Sigma i a_i = \alpha, \quad a_i \geqslant 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (24).$$

The number of terms of (20) corresponding to this partition will be

$$\frac{N!}{a_0!\,a_1!\ldots}.$$

Hence we have

$$M_\alpha = \frac{1}{N^\alpha} \Sigma \frac{N!}{a_0!\,a_1!\ldots} \frac{\alpha!}{(2!)^{a_2}(3!)^{a_3}\ldots} \mu_0^{a_0} \mu_1^{a_1} \mu_2^{a_2} \ldots \quad \ldots\ldots\ldots\ldots (25),$$

the summation being now extended to all possible partitions of (24). For example, the first few cases may be written down (remembering $\mu_1 = 0$) as follows:

$$\left.\begin{aligned} M_1 &= 0 \\ M_2 &= \frac{\mu_2}{N} \\ M_3 &= \frac{\mu_3}{N^2} \\ M_4 &= \frac{1}{N^3}[\mu_4 + 3(N-1)\mu_2^2] \\ M_5 &= \frac{1}{N^4}[\mu_5 + 10(N-1)\mu_3\mu_2] \\ M_6 &= \frac{1}{N^5}[\mu_6 + 15(N-1)\mu_4\mu_2 + 10(N-1)\mu_3^2 + 15(N-1)(N-2)\mu_2^3] \end{aligned}\right\} \ldots (26).$$

* That (17) and (23) are consistent may be seen, e.g. by consulting Nörlund's *Differenzenrechnung* and comparing the sixth equation of p. 139 with the last equation of the same section, on p. 140. The actual values of the $\bar{M}_{2\alpha}$ for the first few values of $\alpha$ may be read off from Nörlund's table 6, p. 460, remembering that in his notation, $\bar{M}_{2\alpha} = \frac{D_{2\alpha}^{(-N)}}{(2N)^{2\alpha}}$. Thus e.g. $\bar{M}_2 = \frac{1}{12N}$, as follows also from (26) and (22).

Introducing the $\beta$'s by the formulæ

$$\beta_{2s-2} = \frac{\mu_{2s}}{\mu_2^s}, \quad \beta_{2s-1} = \frac{\mu_{2s+1}\mu_3}{\mu_2^{s+2}} \quad \text{.....(27),}$$

and similarly for the $B$'s in terms of the $M$'s, we have

$$\left.\begin{aligned} B_1 &= \frac{\beta_1}{N} \\ B_2 - 3 &= \frac{\beta_2 - 3}{N} \\ B_3 - 10B_1 &= \frac{\beta_3 - 10\beta_1}{N^2} \\ B_4 - 15B_2 - 10B_1 + 30 &= \frac{\beta_4 - 15\beta_2 - 10\beta_1 + 30}{N^2} \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \quad \text{.....(28).}$$

As $N \to \infty$, the dominant term in (25) is that which corresponds to a partition having the smallest possible value for $a_0$. This is the partition (2 2 ...) if $\alpha$ is even and (3 2 2 ...) if $\alpha$ is odd. Hence, we have as $N \to \infty$

$$M_{2\alpha} \sim \frac{1}{N^{2\alpha}} \frac{(2\alpha)!}{(2!)^\alpha} \frac{N!}{\alpha!\,(N-\alpha)!} \mu_2^\alpha,$$

or
$$M_{2\alpha} \sim (2\alpha-1)(2\alpha-3)\ldots 5 \cdot 3 \left(\frac{\mu_2}{N}\right)^\alpha = \prod_{r=1}^{\alpha} \{(2r-1)M_2\} \quad \text{.....(29),}$$

while
$$M_{2\alpha+1} \sim \frac{1}{N^{2\alpha+1}} \frac{(2\alpha+1)!}{(2!)^{\alpha-1}\,3!} \frac{N!}{(N-\alpha)!\,(\alpha-1)!} \mu_2^{\alpha-1}\mu_3,$$

or
$$M_{2\alpha+1} \sim \tfrac{1}{3}\alpha(2\alpha+1)(2\alpha-1)(2\alpha-3)\ldots 5 \cdot 3 \frac{\mu_2^{\alpha-1}\mu_3}{N^{\alpha+1}} \quad \text{.....(30).}$$

Whence
$$B_{2\alpha-2} \sim (2\alpha-1)(2\alpha-3)\ldots 7 \cdot 5 \cdot 3 \quad \text{.....(31),}$$

$$B_{2\alpha-3} \sim \tfrac{1}{3}(\alpha-1)(2\alpha-1)(2\alpha-3)\ldots 7 \cdot 5 \cdot 3 \frac{\beta_1}{N} = \tfrac{1}{3}(\alpha-1) B_1 B_{2\alpha-2} \quad \text{...(32).}$$

If we change our unit of length in the ratio $1 : \sqrt{N}$*, we see at once from these equations that the new moments tend to normal-curve values as $N \to \infty$.

### Editorial Note.

Proceeding by completely different methods, Irwin and Hall have obtained equations for the frequency distribution of means in samples from a population following a "rectangular" law of distribution (p. 236, (19) and p. 241, (12)). Their results are of considerable interest, not only in themselves, but for the light they throw upon the distribution of moments in samples from populations in which the variable lies within a finite or limited range.

Hall makes use of a conception which has been employed in connection with the sampling distribution of means and standard deviations from a Normal population†,

* So as to maintain a constant standard deviation as $N$ increases.

† R. A. Fisher, *Biometrika*, Vol. x. p. 507.

representing the sample of $N$ by a point $(x_1, x_2, \ldots, x_N)$ in an $N$-dimensional density space. For a population of limited range, obeying a law $y = f(x)$, we have, choosing suitable units, $0 \leqslant x \leqslant 1$, so that the appropriate density field will be the unit hypercube, with point density at $(x_1, x_2, \ldots, x_N)$ of $f(x_1)\, f(x_2) \ldots f(x_N)$. In the case of the "rectangular" population for which $f(x)$ is constant, the density is uniform throughout the hypercube, but for other forms of distribution, as for example that of the general Type I curve, this will not be so. The frequency distribution of means, $m$, in samples of $N$, will still however be obtained by integrating the density throughout the $N-1$-dimensioned region within the hypercube lying at right angles to the diagonal axis $x_1 = x_2 = \ldots = x_N$. Just as for Hall's simple case, it would appear that the law representing the content of this region will change its form at the $N-1$ points $m = \kappa/N$ $(\kappa = 1, 2, \ldots, N-1)$ so that the resulting distribution of means will consist of $N$ connected arcs.

In the simple case of $N = 3$, the position can be grasped visually from Hall's diagram (p. 240). The frequency of $m$ is here proportional to the integral of a density $f(x_1)\, f(x_2)\, f(x_3)$ over the plane perpendicular to the diagonal $OI$. This plane is represented as cutting $OI$ in $M$ where $OM/OI = m$. As $M$ moves away from $O$ the law of distribution remains of the same form until $OM = \frac{1}{3} OI$ when the section of the plane by the cube changes from a triangle to a hexagon. The new law holds until $OM = \frac{2}{3} OI$, when the section changes back from a hexagon to a triangle, and the third law holds until $M$ reaches $I$ and $m = 1$. For the special case of the "rectangular" population, Irwin deduces these three laws from the general equations for $N = 2$, 3 and 4 on pp. 237—238.

It seems probable that the solution outlined by Irwin at the end of his paper, for Type II curves, will lead also to a system of separate arcs, as the consideration of the hypercube suggests.

The case of the Type III curve is of interest as forming a transition type; one limit of range is removed to infinity, so that only one corner of the hypercube lies at a finite point. As $M$ moves away from $O$ along the diagonal (Hall's figure) the shape of the region lying at right angles to this line, throughout which the density is to be integrated, does not change as long as $m$ remains finite. Consequently there is a *single* law of distribution of means, as given first by Church and later by Irwin (p. 229, (7)). This should also be true for Type VI curves (limited in one direction), as well as for Type IV where the density field will fill the whole of the $N$-dimensional space, as for the Normal curve.

A consideration of the hypercube also suggests that the distribution of standard deviations in samples from a limited range population will also consist of connected arcs. With the lettering of Hall's figure, the standard deviation of the sample is $\sigma = MP/OI$, and the frequency of a given value of $\sigma$ will be obtained by integrating the density throughout the region for which $MP$ is constant, which may be described as a "hypercylinder," with axis $OI$, and cross-section an $(N-1)$-dimensioned hypersphere. As $MP$ increases the law representing the form of this region will change according to the number of faces of the hypercube which it cuts. E. S. P.

# THE MATHEMATICS OF INTELLIGENCE.

## I. THE SAMPLING ERRORS IN THE THEORY OF A GENERALISED FACTOR.

By KARL PEARSON and MARGARET MOUL.

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### (1) *Introductory.*

In a paper under the above title by Professor Spearman and Dr Holzinger in *The British Journal of Psychology** various approximations are given for the probable error of a "tetrad"† of correlation coefficients which we think require a little ampler consideration before they are adopted into psychological practice. The need for a reconsideration of the matter is the more essential because Professor Spearman writes in his recent work, *The Abilities of Man*, as if his mathematical theory was beyond impeachment and his experimental data justified his theory:

"Even with the degree of exactness attained already, however, the agreement of the observed values with those required by theory must be admitted by any unbiased person to have been surprisingly close. In general, it seems quite as good as, if not better than, that usually reached in determining the mechanical equivalent of heat and thus establishing the law of conservation of energy."

The matter is of special interest to one of the present writers, because formulae are cited for which he is largely responsible, and the exact limitation of which appears to be overlooked.

* Vol. xv. pp. 17—19, 1925; *The Abilities of Man*, p. 160.

† There is no reason why the term "tetrad" should not be used for brevity's sake instead of "tetrad difference" for the cross product of the four correlations.

(2) *Nature of Problem.*

We will suppose that $\rho_{st}$ represents the correlation between the $s$th and $t$th variates in an indefinitely large population which is going to be sampled; that $r_{st}$ is the corresponding correlation in any particular sample, and that $\bar{r}_{st}$ is the mean value of $r_{st}$ for many samples. Then $\bar{r}_{st}$ is *not* equal to $\rho_{st}$ and, unless this point is borne in mind, it is not possible to deduce the accurate value to higher approximations of the probable error of functions of the correlation coefficients. Actually we do not know the relation between $\bar{r}_{st}$ and $\rho_{st}$ unless we make some hypothesis as to the frequency surface of the variates under discussion. The only frequency surface for which the relation between $\bar{r}_{st}$ and $\rho_{st}$ has been fully determined is the normal surface. In this case the full expression for $\bar{r}_{st}$ in terms of $\rho_{st}$ will be found in *Biometrika*, Vol. XI. p. 336. An approximate expression, namely,

$$\bar{r}_{st} = \rho_{st}\left[1 - \frac{1-\rho^2_{st}}{2n} + (\ldots)\frac{1}{n^2} + \ldots\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

was given earlier by Mr Soper in the same Journal, Vol. IX. p. 105, where $n$ is the size of the sample. Now Spearman and Holzinger apparently measure their $\delta r_{st}$ from $\rho_{st}$, the value in the entire or sampled population*, but they put the mean values of quantities like $\delta r_{st}$, which we will represent by $\{\delta r_{st}\}$, zero, or they tacitly assume that $\delta r_{st}$ is measured from its mean. In doing this they are neglecting the difference between $\bar{r}_{st}$ and $\rho_{st}$, or terms of the order $1/n$ as compared with 1. Further, the Pearson-Filon formulae† which they cite and use are only correct for $\{\delta r_{st}\delta r_{uv}\}$ and $\{\delta r^2_{st}\}$, (i) when these differentials are measured from their means in samples, (ii) when we suppose normal distribution of the variates, and (iii) when we neglect terms in $1/n^2$ as compared with $1/n$. If we are going to proceed to terms in $1/n^2$ as well as those in $1/n$, then we cannot, as the above authors have done, put $\{\delta r_{st}\}$ zero, nor apply the Pearson-Filon formulae to determine the value of $\{\delta r_{st}\delta r_{uv}\}$. Still less can we assume, knowing that the distribution of $r_{st}$ and $r_{uv}$ is *not* normal, that their combined distribution will be normal, even when we retain terms of the order $1/n^2$. Accordingly we may safely say that their determination of the terms in $1/n^2$ is in error, and we need not enter into a discussion of their attempt to improve their formula involving only terms in $1/n$ by introducing others in $1/n^2$. To obtain such terms correctly we should want to know at least to a second approximation many things of which we are at present ignorant. We can therefore rest content with the statement that we must dismiss as erroneous the formula professing to give terms in $1/n^2$ which Holzinger and Spearman provide‡. But if we could find the correct term in $1/n^2$—which at present we cannot—would it be worth finding? We answer, certainly not, for our authors replace the true values of the correlations, $\rho$, by the observed values, $r$. Now the variation of $r$ round $\rho$ due to random sampling is of the order of its standard deviation, namely $1/\sqrt{n}$, and therefore while terms of the order $1/(n\sqrt{n})$ are neglected, it is idle to attempt the valuation of terms of the order $1/n^2$.

* *loc. cit.* pp. 17—18.

† *Phil. Trans.* Vol. 191 A, p. 262.

‡ *British Journal of Psychology*, Vol. XV. pp. 18—19.

(3) *Variability of a Tetrad.*

Bearing these points in mind let us now turn to the first approximation to the variability of a tetrad. It is needful to find the relations between the various forms of tetrad that may appear and to adopt a notation to distinguish them. In an individual sample we have the tetrad

$$ {}_rT_{st|uv} = r_{su}r_{tv} - r_{tu}r_{sv} \quad \text{.....(ii),} $$

the correlations being those of the sample. When we may without confusion in the text drop the subscript ${}_{st|uv}$, we can write this ${}_rT$ simply. Again

$$ {}_\rho T_{st|uv} = \rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} \quad \text{.....(iii)} $$

will represent the corresponding tetrad in the sampled population.

If $\bar{r}_{su}$, $\bar{r}_{tv}$, $\bar{r}_{tu}$, $\bar{r}_{sv}$ be the mean correlation coefficients in samples we shall require

$$ {}_{\bar{r}}T_{st|uv} = \bar{r}_{su}\bar{r}_{tv} - \bar{r}_{tu}\bar{r}_{sv} \quad \text{.....(iv),} $$

or the tetrad with the mean values of the correlation coefficients in samples.

Finally we may need the mean value of the tetrad in samples $\overline{T}_{st|uv}$, which is not the same thing as ${}_{\bar{r}}T_{st|uv}$. All these various values must be carefully distinguished and interrelated, if we require in the case of small samples to proceed to an approximation involving $1/n^2$ as well as $1/n$.

In order to find the standard deviation of a tetrad, it is needful to measure its variation from $\overline{T}$, its mean value in samples, and not from its value in the sampled population. In order to apply the known results for $\{\delta r^2_{st}\}$ and $\{\delta r_{st}\delta r_{uv}\}$ it is necessary to measure the variations of $r_{st}$, $r_{uv}$, etc. from their mean values in samples. Accordingly we have:

$$ \begin{aligned} {}_rT_{st|uv} &= \overline{T}_{st|uv} + \delta T_{st|uv} = (\bar{r}_{su} + \delta r_{su})(\bar{r}_{tv} + \delta r_{tv}) - (\bar{r}_{tu} + \delta r_{tu})(\bar{r}_{sv} + \delta r_{sv}) \\ &= {}_{\bar{r}}T_{st|uv} + \bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv} + \delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv} \quad \text{.....(v).} \end{aligned} $$

Now if we take the mean value of both sides,

$$ \{\delta T_{st|uv}\} = 0, \quad \{\delta r_{su}\} = 0, \quad \{\delta r_{tv}\} = 0, \quad \{\delta r_{tu}\} = 0, \quad \{\delta r_{sv}\} = 0, $$

because the variations are all measured from their mean values. Accordingly

$$ \overline{T}_{st|uv} = {}_{\bar{r}}T_{st|uv} + \{\delta r_{su}\delta r_{tv}\} - \{\delta r_{tu}\delta r_{sv}\} \quad \text{.....(vi).} $$

To the latter two mean products the Pearson-Filon formulae will apply as the correlation coefficient variations are measured from their means. With regard to the first term

$$ \begin{aligned} {}_{\bar{r}}T_{st|uv} &= \bar{r}_{su}\bar{r}_{tv} - \bar{r}_{tu}\bar{r}_{sv} \\ &= \rho_{su}\rho_{tv}\left(1 - \frac{1-\rho^2_{su}}{2n}\right)\left(1 - \frac{1-\rho^2_{tv}}{2n}\right) - \rho_{tu}\rho_{sv}\left(1 - \frac{1-\rho^2_{tu}}{2n}\right)\left(1 - \frac{1-\rho^2_{sv}}{2n}\right) \\ &= {}_\rho T_{st|uv}\left(1 - \frac{1}{n}\right) + \frac{\rho_{su}\rho_{tv}(\rho^2_{su} + \rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu} + \rho^2_{sv})}{2n} \quad \text{.....(vii),} \end{aligned} $$

using Soper's formula to a first approximation and retaining only terms of order $1/n$. It will be clear accordingly that ${}_{\bar{r}}T_{st|uv}$ is not equal to ${}_\rho T_{st|uv}$, and this distinction must be preserved when we proceed to second order approximations.

We next require $\{\delta r_{su}\delta r_{tv}\}-\{\delta r_{tu}\delta r_{sv}\}$. The complete value of $\{\delta r_{su}\delta r_{tv}\}$ is unknown to us. If we neglect terms of the order $1/n^2$ as compared to $1/n$, which in practice amounts to using only large samples, then we may apply the Pearson-Filon formula, using either $\bar{r}$'s or $\rho$'s, i.e. we can put

$$\{\delta r_{su}\delta r_{tv}\}=\frac{\left\{\begin{array}{l}(\rho_{st}-\rho_{su}\rho_{tu})(\rho_{uv}-\rho_{ut}\rho_{vt})+(\rho_{sv}-\rho_{st}\rho_{vt})(\rho_{ut}-\rho_{su}\rho_{st})\\ +(\rho_{st}-\rho_{sv}\rho_{tv})(\rho_{uv}-\rho_{su}\rho_{sv})+(\rho_{sv}-\rho_{su}\rho_{vu})(\rho_{ut}-\rho_{uv}\rho_{tv})\end{array}\right.}{2n}$$

$$\{\delta r_{tu}\delta r_{sv}\}=\frac{\left\{\begin{array}{l}(\rho_{st}-\rho_{tu}\rho_{su})(\rho_{uv}-\rho_{us}\rho_{vs})+(\rho_{tv}-\rho_{st}\rho_{vs})(\rho_{us}-\rho_{ut}\rho_{st})\\ +(\rho_{st}-\rho_{tv}\rho_{sv})(\rho_{uv}-\rho_{tu}\rho_{tv})+(\rho_{tv}-\rho_{tu}\rho_{vu})(\rho_{us}-\rho_{uv}\rho_{sv})\end{array}\right.}{2n}$$

.....(viii).

Expanding the right-hand sides, subtracting and cancelling, we find:

$$\{\delta r_{su}\delta r_{tv}\}-\{\delta r_{tu}\delta r_{sv}\}={}_{\rho}T_{st|uv}\,(S(\rho^2)-2)/(2n)$$
$$-[\rho_{su}\rho_{tv}(\rho^2{}_{su}+\rho^2{}_{tv})-\rho_{tu}\rho_{sv}(\rho^2{}_{tu}+\rho^2{}_{sv})]/(2n),$$

where $$S(\rho^2)=\rho^2{}_{st}+\rho^2{}_{uv}+\rho^2{}_{su}+\rho^2{}_{tv}+\rho^2{}_{sv}+\rho^2{}_{tu}.$$

Hence by (vi) and (vii)

$$\overline{T}_{st|uv}={}_{\rho}T_{st|uv}\left(1+\frac{S(\rho^2)-4}{2n}\right)\quad\text{.....(ix).}$$

Accordingly we conclude that in general $\overline{T}$, ${}_{\rho}T$ and ${}_{\bar{r}}T$ are not equal, and that in the special case when the tetrad vanishes for the sampled population, although $\overline{T}$ will then be zero, ${}_{\bar{r}}T$ will not be zero unless we can neglect terms of the order $1/n$ as compared with unity. Thus the problem of determining the probable error of $T$ does not reduce to finding merely the third and fourth order mean products, but to something even more fundamental, namely, to the consideration of what $\delta T$ and the $\delta r$'s are measured from, as well as the ascertaining of $\{\delta r^2\}$ and $\{\delta r_{st}\delta r_{uv}\}$, etc. to the $1/n^2$ order. It is not possible to put $\bar{r}_{su}\bar{r}_{tv}-\bar{r}_{tu}\bar{r}_{sv}$ zero, because $\rho_{su}\rho_{tv}-\rho_{tu}\rho_{sv}$ is zero, if one desires to retain higher terms. We need not stay to criticise the Spearman-Holzinger treatment* of the higher terms; they really assume $\rho^2{}_{su}+\rho^2{}_{tv}=\rho^2{}_{tu}+\rho^2{}_{sv}$ for every four variates, which is exceedingly unlikely to be true, nor do they realise that their standard deviation, if true, would be about the mean value in samples and not about the population value. Thus the Spearman-Holzinger equation for $\sigma_{T^2}$, i.e. $\sigma_{F^2}$ in their notation, is in error, and their attempt to allow for terms in $1/n^2$ (their $1/N^2$) fails. Let us now return to their fundamental formula which gives the standard deviation of the tetrad to a first approximation.

(4) *Standard Deviation of a Tetrad.*

Writing $\omega$ for "of order" we have from (vii)

$${}_{\bar{r}}T_{st|uv}={}_{\rho}T_{st|uv}\left(1-\frac{1}{n}\right)+\omega\left(\frac{1}{n}\right),$$

and then from (ix)

$${}_{\bar{r}}T_{st|uv}=\overline{T}_{st|uv}\left(1-\frac{1}{n}-\frac{S(\rho^2)-4}{2n}\right)+\omega\left(\frac{1}{n}\right)$$
$$=\overline{T}_{st|uv}\left(1-\frac{S(\rho^2)-2}{2n}\right)+\omega\left(\frac{1}{n}\right)\quad\text{.....(x).}$$

* *British Journal of Psychology*, Vol. XVI. pp. 87—88.

Accordingly from (v) we have, $\delta T_{st|uv}$ being measured from its true mean value for samples,

$$\delta T_{st|uv} = -\ T_{st|uv}\frac{S(\rho^2)-2}{2n} + \omega\left(\frac{1}{n}\right) + \text{etc.}$$

$$= -\ {}_\rho T_{st|uv}\frac{S(\rho^2)-2}{2n} + \omega\left(\frac{1}{n}\right) + \text{etc.},$$

or, if ${}_\rho T_{st|uv} = 0$ *, then we have, replacing $\omega$,

$$\delta T_{st|uv} = \frac{\rho_{su}\rho_{tv}(\rho^2_{su}+\rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu}+\rho^2_{sv})}{2n}$$
$$+ \bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv}$$
$$+ \delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv} \quad \text{.....(xi).}$$

Now when we square $\delta T$ and take its mean value, the linear terms, whatever the order of approximation we go to, will always vanish, being measured from their means, but no other terms disappear absolutely. Writing

$$\delta T = A + B + C,$$

where $A$, $B$, $C$ correspond to the three lines of (xi), we have:

$$\sigma_T{}^2 = \{A^2\} + 2\{AB\} + 2\{AC\} + \{B^2\} + 2\{BC\} + \{C^2\} \quad \text{.....(xii),}$$

and we note $\{A^2\}$ is of order $1/n^2$, $\{AB\}$ vanishes, $\{AC\}$ is of order $1/n^2$. Thus it follows that $A$ must be retained if we wish to go to order $1/n^2$, but may be dropped if we only wish to go to order $1/n$. Now consider $\{B^2\}$; this involves terms of the forms $\bar{r}^2_{su}\{\delta r^2_{tv}\}$ and $\bar{r}_{su}\bar{r}_{tv}\{\delta r_{tv}\delta r_{su}\}$. If we wish to go only to terms in $1/n$, it will be sufficient to write $\bar{r}_{su} = \rho_{su}$, etc., $\{\delta r^2_{tv}\} = \dfrac{(1-\rho^2_{tv})^2}{n}$ and for products like $\{\delta r_{tv}\delta r_{su}\}$, the values given by (viii). But should we wish to go to terms in $1/n^2$ we must write

$$\bar{r}_{su} = \rho_{su}\left(1 - \frac{1-\rho^2_{st}}{2n}\right)$$

and for $\{\delta r^2_{tv}\}$ we must use the expansions provided in *Biometrika*, Vol. XI. pp. 335—6 (xx*bis*) and (xxv). But a new difficulty arises with regard to $\{\delta r_{tv}\delta r_{su}\}$; we know this to the order $1/n$, but as far as we are aware nobody has yet deduced it to the order $1/n^2$. Until this is done we cannot hope to reach Spearman and Holzinger's desired goal of including terms in $1/n^2$. Next we have threefold products such as $\{\delta r_{su}\delta r^2_{sv}\}$ and $\{\delta r_{su}\delta r_{tv}\delta r_{tu}\}$; these we should anticipate would be of the order $1/n^2$. If we assume the regression linear, the first mean will depend upon $\{\delta r^3_{sv}\}$, or on the $\beta_1$ of the distribution of $r_{sv}$. The wide divergence of the distribution of $r_{sv}$ from a normal curve even for a sample of 25, if $r_{sv}$ is ·5 or over, is well known†. Again some terms in the fourfold products will be of order $1/n^2$. But what their form is we do not yet know.

* If ${}_\rho T_{st|uv}$ be not zero, there will be an additional term in $1/n$, but this only modifies $A$ (see below) and $\sigma_T{}^2$ will still depend on $\{B^2\}$ to a first approximation.

† *Biometrika*, Vol. XI. pp. 396—403, for values of $\beta_1$ and $\beta_2$.

Finally we would suggest that it is impossible to solve Spearman and Holzinger's problem of the expansion of $\sigma_T{}^2$ to terms in $1/n^2$ as well as $1/n$ in order to meet the case of small samples, until the correlation surface of correlation coefficients is known, which it is not at present. Even then we would point out two grave difficulties which arise, namely (*a*) all the results used apply only to sampling from normal populations, and the errors introduced by the grade of anormality of the distribution of many mental characters will certainly exceed anything capable of being corrected by using additional terms in $1/n^2$. But the second difficulty is still graver: (*b*) What values do Spearman and Holzinger propose to use for the population values $\rho$? Presumably they use those of the individual sample, which is all they have knowledge of. But, as we have already pointed out, these values differ from the true or population values by terms of the order $\sigma_r$, i.e. by terms of the order $1/\sqrt{n}$. It is therefore idle to retain terms of the order $1/n^2$, when we neglect terms of the order $1/n \times 1/\sqrt{n}$, which occur in connection with replacing $\rho_{st}$ by $r_{st}$ in the terms of order $1/n$. The terms in $1/n^2$ would only be of value provided we were dealing with small sampling from a population with *a priori* known values of the $\rho$'s.

Clearly if we wish to obtain $\sigma_T$ to the order $1/n$ only, it will be adequate to retain only the $B^2$ terms, and this is precisely what Spearman and Holzinger have done without adequately describing why they neglect the $A$ terms, and that inadequacy causes them to overlook points necessary to establish their second result.

Remembering that if we neglect terms in $1/n$ as compared with unity, $\bar{r}_{st}$ may be replaced by $\rho_{st}$, we easily find for $\sigma_{T_{st|uv}}$:

$$\begin{aligned}\sigma^2_{T_{st|uv}} = 1/n\,[&\rho^2_{tv}(1-\rho^2_{su}) + \rho^2_{su}(1-\rho^2_{tv}) + \rho^2_{sv}(1-\rho^2_{tu}) + \rho^2_{tu}(1-\rho^2_{sv}) \\ &+ \rho_{tv}\rho_{su}Q_{su,tv} + \rho_{sv}\rho_{tu}Q_{tu,sv} - \rho_{tv}\rho_{sv}Q_{su,tu} \\ &- \rho_{tv}\rho_{tu}Q_{su,sv} - \rho_{su}\rho_{sv}Q_{tv,tu} - \rho_{su}\rho_{tu}Q_{tv,sv}] \quad \text{.....(xiii),}\end{aligned}$$

where $Q_{su,tv}$, etc. are the numerators in the Pearson-Filon formulae for the mean values $\{\delta r_{su}\delta r_{tv}\}$, etc. given by Equation (viii) above.

If now we substitute the $\rho$ values for the $Q$'s we obtain after very lengthy but straightforward algebra—which it seems unnecessary to reproduce here—and on rearrangement:

$$\begin{aligned}\sigma^2_{T_{st|uv}} = 1/n\,[&\rho^2_{su} + \rho^2_{sv} + \rho^2_{tu} + \rho^2_{tv} \\ &- 2(\rho_{st}\rho_{su}\rho_{tu} + \rho_{st}\rho_{sv}\rho_{tv} + \rho_{su}\rho_{sv}\rho_{uv} + \rho_{tv}\rho_{tu}\rho_{uv}) \\ &+ 2\rho_{st}\rho_{uv}(\rho_{su}\rho_{tv} + \rho_{sv}\rho_{tu}) \\ &+ (\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv})^2(S(\rho^2) - 4)] \quad \text{.....(xiv).}\end{aligned}$$

If the tetrad $T_{st|uv}$ be really zero in the sampled population the last term on the fourth line vanishes, and our result is then in entire agreement with the Spearman-Holzinger formula for the terms in $1/n$. Further, Equation (ix) shows us that while $T_{st|uv}$—the mean about which $\sigma_{T_{st|uv}}$ is taken—is not equal to ${}_\rho T_{st|uv}$ in general, yet, in the particular case of the tetrad vanishing, $\overline{T}_{st|uv}$ does equal ${}_\rho T_{st|uv}$, because both vanish together. It thus seems to us that it is necessary to establish (ix) before we can show that $\sigma_{T_{st|uv}}$ is really taken about the zero value of the sampled

population. Actually Equations (ix) and (xiv) give the mean and variance of a tetrad whether it is or it is not zero in the sampled population*.

Now consider the four variates $s$, $t$, $u$, $v$ leading to the six correlation coefficients

$$\rho_{st},\ \rho_{uv},\ \rho_{su},\ \rho_{sv},\ \rho_{tu},\ \rho_{tv}$$

connected with the tetrad $T_{st|uv}$; they are equally connected with the two further tetrads $T_{tu|vs}$ and $T_{us|vt}$. We have in fact:

$$\left.\begin{aligned} T_{st|uv} &= \rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} \\ T_{tu|vs} &= \rho_{ts}\rho_{uv} - \rho_{tv}\rho_{us} \\ T_{us|vt} &= \rho_{ut}\rho_{vs} - \rho_{uv}\rho_{st} \end{aligned}\right\} \quad \text{.....(xv).}$$

We shall speak of these tetrads which arise from the four variates as a triplet of tetrads. We see accordingly that whether the tetrads do or do not vanish, if the correlation products on the right are taken in cyclical order,

$$T_{st|uv} + T_{tu|vs} + T_{us|vt} = 0,$$

or in a long series of tetrads each triplet sum if taken in proper order will vanish, or the mean tetrad of a system of tetrads will be actually zero, if the triplets are properly taken. If not taken in cyclical order, but at random without regard to sign, then the mean tetrad of a system will be very nearly zero. In other words: *The vanishing of the mean tetrad is no proof whatever of the individual tetrads being zero or small; it is merely a result of their method of construction and the interrelation of the tetrads in triplets.*

(5) *Approximation to Mean Square Tetrad.*

Now let us consider the mean squared standard deviation of a triplet of tetrads. Clearly from (xiv) we have:

$$\begin{aligned} &\tfrac{1}{3}\left[\sigma^2_{T_{st|uv}} + \sigma^2_{T_{tu|vs}} + \sigma^2_{T_{us|vt}}\right] \\ &\quad = 1/n\left[\tfrac{2}{3}S(\rho^2) - 2\left(\rho_{st}\rho_{su}\rho_{tu} + \rho_{st}\rho_{tv}\rho_{vs} + \rho_{su}\rho_{uv}\rho_{vs} + \rho_{tu}\rho_{uv}\rho_{vt}\right)\right. \\ &\qquad + \tfrac{4}{3}\left(\rho_{st}\rho_{uv}\rho_{us}\rho_{tv} + \rho_{us}\rho_{tv}\rho_{tu}\rho_{sv} + \rho_{tu}\rho_{sv}\rho_{st}\rho_{uv}\right) \\ &\qquad \left. + \tfrac{1}{3}\left((\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv})^2 + (\rho_{ts}\rho_{uv} - \rho_{tv}\rho_{us})^2 + (\rho_{tu}\rho_{sv} - \rho_{uv}\rho_{st})^2\right) \times \left(S(\rho^2) - 4\right)\right] \\ &\qquad\qquad \text{.....(xvi).} \end{aligned}$$

Here the last line will be zero if the tetrads vanish in the sampled population.

To obtain the squared standard deviation of all the tetrads we have to sum (xvi) for all triplets and divide by their number, $\frac{1}{8}m(m-1)(m-2)(m-3)$.

Now although there is nothing more to hinder the calculation of the expression in (xvi) for a hundred or a thousand triplets of tetrads† than the immense labour of the arithmetic, it would undoubtedly be a gain could we throw back the determination of the mean square tetrad on constants of the original distribution of

* This follows because the term ${}_\rho T_{st|uv}(S(\rho^2) - 2)$ only adds to the $A$ or $1/n$ terms in (xi), which disappear on taking the variance to a first approximation. Of course, when ${}_\rho T_{st|uv}$ is not zero, the variance is measured round the value of ${}_\rho T_{st|uv}$ to a first approximation and not round zero.

† We have actually carried this out in the case of the 378 tetrads of Holzinger's data and the 3003 tetrads of Simpson's data.

correlation coefficients. Holzinger and Spearman have attempted this by a method which we shall generalise in this section of our discussion. We beg the reader's careful attention for this analysis, as, though it appears plausible enough, we fear that it is really erroneous. First a preliminary proposition: *To determine the sums of products of variates* $X_1, X_2, \ldots X_m$ *for which* $X_1 + X_2 + \ldots + X_m = 0$.

Since $X_1 + X_2 + \ldots + X_m = 0$,

$$S(X_1^2) + 2S(X_1X_2) = 0.$$

Now let curled brackets denote, as before, the mean of the sum of a quantity contained in them. Then the above equation becomes

$$m\{X_1^2\} = -2\frac{m(m-1)}{2}\{X_1X_2\}.$$

We shall write $\{X_1^s\}^2 = \mu_s$, and accordingly

$$\{X_1X_2\} = -\mu_2/(m-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xvii)}.$$

Again: $(X_1 + X_2 + \ldots + X_m)(X_1^2 + X_2^2 + \ldots + X_m^2) = 0,$

$$S(X_1^3) + S(X_1^2X_2) = 0,$$

$$m\mu_3 + m(m-1)\{X_1^2X_2\} = 0,$$

$$\{X_1^2X_2\} = -\mu_3/(m-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xviii)}.$$

Further: $(X_1 + X_2 + \ldots + X_m)^3 = 0,$

therefore: $S(X_1^3) + 3S(X_1^2X_2) + 6S(X_1X_2X_3) = 0,$

or $2S(X_1^3) = 6S(X_1X_2X_3),$

$$m\mu_3 = \frac{3m(m-1)(m-2)}{6}\{X_1X_2X_3\},$$

$$\{X_1X_2X_3\} = \frac{2}{(m-1)(m-2)}\mu_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xix)},$$

which completes the third order products. Again:

$$(X_1 + X_2 + \ldots + X_m)^4 = 0,$$

$$S(X_1^4) + 4S(X_1^3X_2) + 6S(X_1^2X_2^2) + 12S(X_1^2X_2X_3) + 24S(X_1X_2X_3X_4) = 0.$$

But $(S(X_1^2))^2 = (X_1^2 + X_2^2 + \ldots + X_m^2)^2 = S(X_1^4) + 2S(X_1^2X_2^2),$

or $$m^2\mu_2^2 = m\mu_4 + \frac{2m(m-1)}{2}\{X_1^2X_2^2\};$$

thus: $$\{X_1^2X_2^2\} = \frac{m\mu_2^2 - \mu_4}{m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xx)}.$$

Further: $(X_1^2 + X_2^2 + \ldots + X_m^2)(X_1 + X_2 + \ldots + X_m)^2 = 0,$

$$S(X_1^4) + 2S(X_1^2X_2^2) + 2S(X_1^3X_2) + 2S(X_1^2X_2X_3) = 0,$$

and $$S(X_1^3)(X_1 + X_2 + \ldots + X_m) = 0,$$

$$S(X_1^4) + S(X_1^3X_2) = 0.$$

Accordingly: $$S(X_1^3X_2) = -S(X_1^4),$$

$$S(X_1^2X_2X_3) = S(X_1^4) - \tfrac{1}{2}(S(X_1^2))^2,$$

and $$24S(X_1X_2X_3X_4) = 3(S(X_1^2))^2 - 6S(X_1^4).$$

Thus: $$\{X_1^3X_2\} = -\mu_4/(m-1) \quad \text{(xxi)},$$

$$\{X_1^2X_2X_3\} = -\frac{(m\mu_2^2 - 2\mu_4)}{(m-1)(m-2)} \quad \text{(xxii)},$$

$$\{X_1X_2X_3X_4\} = \frac{3(m\mu_2^2 - 2\mu_4)}{(m-1)(m-2)(m-3)} \quad \text{(xxiii)}.$$

So far so good; we do not think that there is any exception to be taken to these results. Now let us endeavour to apply these results to the mean squared standard deviation of a triplet in (xvi), omitting the term involving the squared tetrads. For brevity let us write:

$$\rho_{su} = \rho_1 = \bar{\rho} + X_1, \quad \rho_{sv} = \rho_2 = \bar{\rho} + X_2, \quad \rho_{st} = \rho_3 = \bar{\rho} + X_3,$$

$$\rho_{tu} = \rho_4 = \bar{\rho} + X_4, \quad \rho_{tv} = \rho_5 = \bar{\rho} + X_5, \quad \rho_{uv} = \rho_6 = \bar{\rho} + X_6,$$

and $\Sigma^2_{stuv}$ for the left-hand side. Then:

$$\begin{aligned}
\Sigma^2_{stuv} = 1/n\,[&\tfrac{2}{3}(6\bar{\rho}^2 + 2\bar{\rho}(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&\qquad + X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2) \\
&- 8\bar{\rho}^3 - 4\bar{\rho}^2(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&- 2\bar{\rho}(X_1X_3 + X_3X_4 + X_4X_1 + X_3X_5 + X_3X_2 + X_2X_5 + X_1X_6 + X_6X_2 \\
&\qquad + X_2X_1 + X_4X_6 + X_6X_5 + X_5X_4) \\
&- 2(X_1X_3X_4 + X_2X_3X_2 + X_1X_6X_2 + X_4X_6X_5) \\
&+ 4\bar{\rho}^4 + \tfrac{8}{3}\bar{\rho}^3(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) \\
&+ \tfrac{4}{3}\bar{\rho}^2(X_1X_3 + X_1X_6 + X_3X_5 + X_5X_6 + X_1X_4 + X_1X_2 + X_4X_5 + X_2X \\
&\qquad + X_4X_3 + X_4X_6 + X_2X_3 + X_2X_6 \\
&\qquad + 2(X_3X_6 + X_1X_5 + X_2X_4)) \\
&+ \tfrac{4}{3}\bar{\rho}(X_3X_6X_1 + X_3X_6X_5 + X_3X_1X_5 + X_6X_1X_5 + X_1X_5X_4 + X_5X_4X_2 \\
&\quad + X_1X_5X_2 + X_1X_4X_2 + X_4X_3X_6 + X_2X_3X_6 + X_4X_2X_6 + X_4X_2X_3) \\
&+ \tfrac{4}{3}(X_3X_6X_1X_5 + X_1X_5X_4X_2 + X_4X_2X_3X_6)] \quad \text{(xxiv)}.
\end{aligned}$$

Now let us suppose, first, that we have only this triplet of tetrads and $\bar{\rho}$ is the mean of the six correlations, then $X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 0$. Also $X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = \sigma_\rho^2$. But the product of the $X$'s two at a time does not contain all the couplets: it has 12 instead of 15, the omitted couplets being $X_1X_5$, $X_2X_4$ and $X_3X_6$; it cannot therefore be represented by $S(X_1X_2)$. Next, the product of the $X$'s three at a time contains only four out of 20 such products. It cannot therefore possibly be represented by $S(X_1X_2X_3)$. Turning to the fourth order terms, the coefficient of $\frac{8}{3}\bar{\rho}^3$ is $S(X_1)$ and vanishes; the coefficient of $\frac{4}{3}\bar{\rho}^2$ contains all the 15 couplets, but three of them occur twice; it cannot therefore be represented by $S(X_1X_2)$. The coefficient of $\frac{4}{3}\rho$ contains only 12 out of the 20 $X$ triplets and again cannot be represented by $S(X_1X_2X_3)$. Lastly, there

are only three quadruplets out of the 15 requisite to form $S(X_1X_2X_3X_4)$. Further, these differences will still persist and to the same degree of inexactitude when we sum all possible triplets of tetrads. It is clear therefore that the process by which Spearman and Holzinger suppose these $X$ products to correspond to $S(X_1X_2)$, $S(X_1X_2X_3)$ and $S(X_1X_2X_3X_4)$ is in error; and it is not easy to say what is the extent of the error. Their method is difficult to follow, but it appears to be of this nature: They take a product in a single triplet, say $X_3X_6X_1X_5$ or $(\rho_{su}-\bar{\rho})(\rho_{tv}-\bar{\rho})(\rho_{st}-\bar{\rho})(\rho_{uv}-\bar{\rho})$, and taking the mean for all tetrads suppose it to be $\{X_1X_2X_3X_4\}$, but this it clearly is not, because 12 products of type, say $(\rho_{su}-\bar{\rho})(\rho_{tu}-\bar{\rho})(\rho_{tv}-\bar{\rho})(\rho_{uv}-\bar{\rho})$, which contribute to form $\{X_1X_2X_3X_4\}$ can never appear in the mean square tetrad expansion, and we have no right to assume the mean without these terms will be the same as the mean including them. It must, we think, be clear that when 12 terms are used instead of 15, four instead of 20 and three instead of 15 to find means, it cannot be done without risk of very considerable error.

If we may assume that the sum of $X_1X_3$ or $X_1X_3X_4$ or $X_3X_6X_1X_5$ throughout the whole series of triplets of tetrads divided by the number of triplets is $\{X_1X_2\}$ or $\{X_1X_2X_3\}$ or $\{X_1X_2X_3X_4\}$—which it clearly is not, for it excludes values essential to the latter—we can easily arrive at an expression for the mean square tetrad in terms of the constants of the distribution of correlation coefficients. We have

$$\{\psi_r^2\} = \sigma_T^2 = 1/n\,[4\bar{\rho}^2(1-\bar{\rho})^2 + 4\{X^2\} - 24\bar{\rho}\{X_1X_2\} - 8\{X_1X_2X_3\} + \tfrac{4}{3}\bar{\rho}^2 . 18\{X_1X_2\} + \tfrac{4}{3}\bar{\rho} . 12\{X_1X_2X_3\} + \tfrac{4}{3} . 3\{X_1X_2X_3X_4\}],$$

or, using our values above for the mean products in (xvii), (xix) and (xxiii),

$$\sigma_T^2 = \frac{4}{n}\left[\bar{\rho}^2(1-\bar{\rho})^2 + \mu_2\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right) - \frac{4(1-2\bar{\rho})\mu_3}{(m-1)(m-2)} + \frac{3(m\mu_2^2 - 2\mu_4)}{(m-1)(m-2)(m-3)}\right] \quad \text{.....(xxv).}$$

There is no difficulty or excessive labour in finding $\bar{r}$, $\mu_2$, $\mu_3$ and $\mu_4$ for the actual distribution of correlation coefficients, but the question arises as to how far we are justified in retaining the terms with denominators in $m-1$, $m-2$ and $m-3$, when (*a*) we are replacing the true $\bar{\rho}$, $\mu_2$, $\mu_3$, $\mu_4$ by the observed values. We have indicated that this is very doubtful, and that the errors so introduced may be of the same order as the retained terms*. (*b*) We have deduced these terms by a fallacious reasoning. It is true that this reasoning is wider of the mark for the terms in which the third and fourth order products occur than for those in which the second order products occur, but it is just possible that with certain distributions they might rise into importance. However, we have not found cases in which they do†. But the case of the second order products is far more serious, arising partly from the fact that certain terms of $S(X_1X_2)$ do not occur, and partly from the fact that those that do occur are repeated a number of times, which the above form of

* See p. 247.

† We have tested on a variety of data, Holzinger's, Simpson's, Macdonell's, etc. They may affect $\sigma_T$ in the sixth decimal place.

reasoning, following Holzinger and Spearman, appears to neglect. Let us consider the matter a little more closely. If $p$ variates be measured, then the number of correlation coefficients is $m = \frac{1}{2}p(p-1)$ and the number of pairs of correlation coefficients is $\frac{1}{2}(\frac{1}{2}p(p-1))(\frac{1}{2}p(p-1)-1) = \frac{1}{8}p(p-1)(p-2)(p+1)$. This represents the number of terms in $S(X_1X_2)$. Now in any triplet of tetrads there are only 12 instead of 15 terms in the coefficient of $2\bar{\rho}$, the three omitted products being those in which the whole four variates occur as subscripts, i.e. such as $X_{st}X_{uv}$. Now there are $\frac{1}{24}p(p-1)(p-2)(p-3)$ triplets, and three such omitted products occur in each triplet; thus we have altogether $\frac{1}{8}p(p-1)(p-2)(p-3)$ omitted products. Now these products never repeat themselves and we will call their sum $\Sigma(X_1X_2)$. Of the non-omitted products, we have

$$\tfrac{1}{8}p(p-1)(p-2)(p+1) - \tfrac{1}{8}p(p-1)(p-2)(p-3) = \tfrac{1}{2}p(p-1)(p-2).$$

But the total number of such products on the right of our tetrad equation is $12 \times$ number of triplets or is $\frac{1}{2}p(p-1)(p-2)(p-3)$. Hence every one of the non-omitted products occurs $(p-3)$ times. This point appears to be overlooked in the Holzinger-Spearman argument. Now let us consider the terms involving the $X_1X_2$ products. They are:

$$\begin{aligned}
&-2\bar{\rho}\ ((p-3)S(X_1X_2) - (p-3)\Sigma(X_1X_2)) \\
&+\tfrac{4}{3}\bar{\rho}^2((p-3)S(X_1X_2) - (p-3)\Sigma(X_1X_2) + 2\Sigma(X_1X_2)) \\
&= -(2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)(p-3)S(X_1X_2) + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\Sigma(X_1X_2) \\
&= -(2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)(p-3)\tfrac{1}{8}p(p-1)(p-2)(p+1)\{X_1X_2\} \\
&\qquad + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\Sigma(X_1X_2) \\
&= (2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)\tfrac{1}{8}(p-3)p(p-2)(p+1)\mu_2 \\
&\qquad + (2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))\tfrac{1}{8}p(p-1)(p-2)(p-3)[X_1X_2],
\end{aligned}$$

where $[X_1X_2]$ stands for the mean value of the omitted products. Now to obtain the mean square tetrad we must divide by the number of triplets, i.e. $\frac{1}{24}p(p-1)(p-2)(p-3)$. We note in doing this that the $X^2$ terms are

$$\tfrac{1}{2}p(p-1) \times \tfrac{1}{2}(p-2)(p-3)$$

in number, for there are $\frac{1}{2}p(p-1)$ correlation coefficients, and any one $X_{st}$ of them occurs in the tetrad $T_{st|uv}$ and $uv$ may take all possible values, i.e.

$$\tfrac{1}{2}(p-2)(p-3).$$

Hence the term is

$$\frac{2}{3}\frac{p(p-1)(p-2)(p-3)}{4}\{X^2\} = \tfrac{1}{6}p(p-1)(p-2)(p-3)\mu_2.$$

Dividing by the number of triplets we have accordingly

$$\begin{aligned}
\sigma_T^2 = \frac{1}{n}(&4\bar{\rho}^2(1-\bar{\rho})^2 + 4\mu_2 + (2\bar{\rho} - \tfrac{4}{3}\bar{\rho}^2)\frac{3p+1}{m-1}\mu_2 \\
&+ 3(2\bar{\rho}(p-3) - \tfrac{4}{3}\bar{\rho}^2(p-5))[X_1X_2]).
\end{aligned}$$

This is, we believe, correct and if the mean of the "omitted" $X_1X_2$ were computed, this should give a close value for the $\sigma_T^2$. But the labour of finding the mean of

two or three thousand products is almost as great as finding the actual value of the mean square tetrad. The only solution is to assume $[X_1X_2]$ will not differ very widely from $\{X_1X_2\}$, i.e. may be put $-\mu_2/(m-1)$. In this case

$$\sigma_T^2 = \frac{1}{n}\left(4\bar{\rho}^2(1-\bar{\rho})^2 + 4\mu_2 + 6\bar{\rho}\mu_2\left(\frac{p+1}{m-1} - \frac{p-3}{m-1}\right) - 4\bar{\rho}^2\mu_2\left(\frac{p+1}{m-1} - \frac{p-5}{m-1}\right)\right)$$

$$= \frac{4}{n}\left(\bar{\rho}^2(1-\bar{\rho})^2 + \mu_2\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)\right) \quad \text{.....(xxvi).}$$

This is in agreement with the value given by Holzinger and Spearman*. The practical limitations of this formula must, however, be recognised. First, we are going to replace the $\rho$'s by the $r$'s; secondly, the term $6\bar{\rho}(1-\bar{\rho})/(m-1)$ is only a fairly rough approximation as we have assumed that $[X_1X_2]$ may be taken equal to $\{X_1X_2\}$, so that the "omitted" products have the mean value $-\mu_2/(m-1)$. Now the $X$'s of the omitted products do have the same mean and the same variance as the total $X$'s, but they do not satisfy the relation $\Sigma(X_1X_2) = -\frac{1}{2}\Sigma(X^2)$ because $\Sigma(X^2)$ is really $S(X^2)$, but $\Sigma(X_1X_2)$ is very far from being $S(X_1X_2)$†. We in fact can only tell the degree of approximation of (xxvi) to the mean square tetrad by actual calculation of the latter. This we have done in several cases which will be referred to later. The mischief is that the deviation of (xxvi) from the true mean square tetrad may be of the order of the probable error of the latter, and this adds to the difficulty of determining whether the observed mean square tetrad is within reasonable distance of the theoretical value.

(6) *Probable Error of the Mean Square Tetrad.*

One use of (xxvi) is, however, possible. It is the mean square tetrad with *some* approximation and may accordingly be used to find the probable error of the mean square tetrad. We cannot be content to write down the observed mean square tetrad and the above theoretical value, without any measure of the probable error of their difference, and say "How near they are!" if we have no measure of the variation of their difference. Thus: ·010 and ·012 may or may not be "near" each other according as the probable error of their difference is of the order ·002 or ·0002.

Taking as an adequate approximation for probable error purposes

$$\sigma_T^2 = \frac{4}{n}\left(\bar{r}^2(1-\bar{r})^2 + \mu_2\left(1 + \frac{6\bar{r}(1-\bar{r})}{m-1}\right)\right),$$

we have

$$\delta\sigma_T^2 = \frac{4}{n}\left(2\bar{\rho}(1-\bar{\rho})(1-2\bar{\rho}) + \frac{6(1-2\bar{\rho})}{m-1}\mu_2\right)\delta\bar{r} + \frac{4}{n}\left(1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1}\right)\delta\mu_2,$$

* *British Journal of Psychology*, Vol. XVI. p. 88. Still another value is suggested by Spearman, *The Abilities of Man* (Appendix, p. xi), but we have not seen his proof of it.

† For example, in Simpson's data $S(X_1X_2)$ contains 4095 products, while $\Sigma(X_1X_2)$ contains only 3003. How far may the mean of the 3003 products be taken equal to that of the 4095 products? It seems impossible to say without experiment.

and accordingly

$$\sigma^2_{\sigma_{T^2}} = \frac{64}{n^2}\left(\bar{\rho}\,(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3\,(1-2\bar{\rho})}{m-1}\,\mu_2\right)^2 \sigma^2_{\bar{r}} + \frac{16}{n^2}\left(1 + \frac{6\bar{\rho}\,(1-\bar{\rho})}{m-1}\right)^2 \sigma^2_{\mu_2}$$
$$+ \frac{64}{n^2}\left(\bar{\rho}\,(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3\,(1-2\bar{\rho})}{m-1}\,\mu_2\right)\left(1 + \frac{6\bar{\rho}\,(1-\bar{\rho})}{m-1}\right)\{\delta\bar{r}\,\delta\mu_2\}$$
.....(xxvii).

If the distribution of $r$'s were normal and without intercorrelation, then there would be no correlation between the mean and variance of $r$, or $\{\delta\bar{r}\,\delta\mu_2\}$ would be zero. Hence approximately

$$\sigma_{\sigma_{T^2}} = \frac{8}{n}\left[\left(\bar{\rho}\,(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3\,(1-2\bar{\rho})}{m-1}\,\mu_2\right)^2 \sigma^2_{\bar{r}} + \frac{1}{4}\left(1 + \frac{6\bar{\rho}\,(1-\bar{\rho})}{m-1}\right)^2 \sigma^2_{\mu_2}\right]^{\frac{1}{2}}$$
.....(xxviii).

Under ordinary circumstances the standard deviation squared of a mean based upon $m$ variants is $\mu_2/m$ and of its variance $(\mu_4 - \mu_2^2)/m = 2\mu_2^2/m$ approximately, and our result would be

$$\sigma_{\sigma_{T^2}} = \frac{8}{n\sqrt{m}}\left[\left(\bar{\rho} + \frac{3\,(1-2\bar{\rho})}{m-1}\,\mu_2\right)^2 \mu_2 + \frac{1}{2}\left(1 + \frac{6\bar{\rho}\,(1-\bar{\rho})}{m-1}\right)^2 \mu_2^2\right]^{\frac{1}{2}},$$

but this gives a value of quite the wrong order, because we have omitted to consider that the $m$ variants on which $\bar{r}$ and $\mu_2$ are based are correlated together. We will accordingly now endeavour to find at least approximate values for $\sigma^2_{\bar{r}}$, $\sigma^2_{\mu_2}$ and $\{\delta\bar{r}\,\delta\mu_2\}$. Let us take the last first in order to reach some measure of its magnitude and how far it may reasonably be neglected. We have

$$\delta\bar{r} = S(\delta r_s)/m, \text{ where } r_s = \rho_s + \delta r_s,$$
$$\delta\mu_2 = \frac{2}{m} S(\rho_s - \bar{\rho})\,\delta r_s - \frac{2}{m}\,\delta\bar{r}\,S(\rho_s - \bar{\rho}),$$

and the last summation vanishes. Accordingly

$$\delta\bar{r}\,\delta\mu_2 = \frac{2}{m^2} S(\delta r_s) \times S(\rho_s - \bar{\rho})\,\delta r_s$$
$$= \frac{2}{m^2} S(\rho_s - \bar{\rho})\,\delta r_s^2 + \frac{2}{m^2} S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\,(\delta r_s \delta r_t),$$

or, taking mean values,

$$\{\delta\bar{r}\,\delta\mu_2\} = \frac{2}{m^2}\left[S(\rho_s - \bar{\rho})\frac{(1-\rho_s^2)^2}{n} + S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\,\{\delta r_s \delta r_t\}\right]$$
$$= \frac{2}{mn}\left[-4\bar{\rho}\,(1-\bar{\rho}^2)\,\mu_3 - 2\,(1-3\bar{\rho}^2)\,\mu_3 + 4\bar{\rho}\mu_4 + \mu_5\right.$$
$$\left. + \frac{S((\rho_s - \bar{\rho}) + (\rho_t - \bar{\rho}))\,\{n\,\delta r_s \delta r_t\}}{m}\right]$$
.....(xxix).

Now it seems probable that the summation term is very small. If we give $\{n\,\delta r_s \delta r_t\}$ its mean value for all correlation pairs and then sum the summation vanishes. If $r_s$ and $r_t$ do not contain any common variate, then $\{\delta r_s \delta r_t\}$ is of the

fourth order in quantities less than unity and these are multiplied by differences of correlation coefficients which are as often positive as negative. When $r_s$ and $r_t$ do contain a common variate, then a portion of $\{\delta r_s \delta r_t\}$ rises to the cubic order in correlation coefficients, or, if we take $\rho_s - \bar{\rho}$ to be of the second order, to the fifth order. Again these terms of the fifth order will be in part negative and in part positive, and we consider that the summation term as well as $\mu_5$ may with high probability be negligible. We have not thus far succeeded in reducing the summation term to a simple expression, nor do we think that its value could be computed without enormous labour from the observations*.

With the above assumptions we may write

$$\{\delta\bar{r}\,\delta\mu_2\} = \frac{2}{mn}[-4\bar{\rho}(1-\bar{\rho}^2)\mu_2 - 2(1-3\bar{\rho}^2)\mu_3 + 4\bar{\rho}\mu_4] \quad \text{.....(xxx).}$$

We will now consider $\sigma_{\bar{r}}^2$. We have, from $\bar{r} = S(r_s)/m$,

$$\delta\bar{r} = \frac{1}{m} S(\delta r_s),$$

and therefore

$$\begin{aligned}
\sigma_{\bar{r}}^2 &= \frac{1}{m^2}\{S(\delta r_s)^2\} + \frac{2}{m^2}\{S(\delta r_s \delta r_t)\} \\
&= \frac{1}{m^2}\left(\frac{S(1-\rho_s^2)^2}{n}\right) + \frac{2}{m^2} S\{\delta r_s \delta r_t\} \\
&= \frac{1}{mn}\left(\frac{S(1-(\rho_s-\bar{\rho}+\bar{\rho})^2)^2}{m} + \frac{2S\{n\,\delta r_s \delta r_t\}}{m}\right) \\
&= \frac{1}{mn}\left((1-\bar{\rho}^2)^2 - 2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4 + \frac{2S(n\{\delta r_s \delta r_t\})}{m}\right) \quad \text{...(xxxi).}
\end{aligned}$$

Again we see the same difficulty arising about the evaluation of the great number of terms in $\{\delta r_s \delta r_t\}$. To appreciate the order of magnitude of this term in relation to the others, we will determine its value if all the correlation coefficients were given the mean value $\bar{\rho}$. Now there are two types of products $\{\delta r_{ab}\delta r_{cd}\}$ and $\{\delta r_{ab}\delta r_{ad}\}$; the former gives rise to the mean value $2\bar{\rho}^2(1-\bar{\rho})^2/n$ and the latter to the mean value $\bar{\rho}(1-\bar{\rho})^2(2+3\bar{\rho})/(2n)$. Of the former there are

$$\tfrac{1}{8}p(p-1)(p-2)(p-3)$$

and of the latter $\frac{1}{2}p(p-1)(p-2)$, the total number being

$$\tfrac{1}{8}p(p-1)(p-2)(p+1) = \tfrac{1}{2}m(m-1),$$

since $m$, the total number of correlation coefficients, $= \frac{1}{2}p(p-1)$, where $p$ is the number of variates. Hence we may take as approximating to the mean value of $\{\delta r_s \delta r_t\}$:

$$\frac{\tfrac{1}{8}p(p-1)(p-2)(p-3)\,2\bar{\rho}^2(1-\bar{\rho})^2/n + \tfrac{1}{2}p(p-1)(p-2)\,\bar{\rho}(1-\bar{\rho})^2(2+3\bar{\rho})/(2n)}{\tfrac{1}{8}p(p-1)(p-2)(p+1)}$$

$$= \frac{2}{n(p+1)}\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}) \quad \text{.....(xxxii).}$$

* Holzinger's data would involve the finding of $((\rho_s-\bar{\rho})+(\rho_t-\bar{\rho}))\{n\delta r_s \delta r_t\}$ 630 times and Simpson's no less than 4095 times!

In $S(\delta r_s \delta r_t)$ there are $\frac{1}{2}m(m-1)$ product terms. Hence

$$\frac{2S(n\delta r_s \delta r_t)}{m} = \frac{2(m-1)}{p+1}\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}) = (p-2)\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho}).$$

Thus approximately*

$$\sigma_{\bar{r}}^2 = \frac{1}{mn}[(1-\bar{\rho}^2)^2 - 2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4 + (p-2)\bar{\rho}(1-\bar{\rho})^2(2+p\bar{\rho})] \quad \text{.....(xxxiii).}$$

We turn lastly to $\sigma^2_{\mu_2}$. We have $\mu_2 = S(r_s - \bar{r})^2/m$,

$$\delta\mu_2 = \frac{2}{m}S(\rho_s - \bar{\rho})\delta r_s - \frac{2}{m}\delta\bar{r}\,S(\rho_s - \bar{\rho}),$$

but $S(\rho_s - \bar{\rho}) = 0$, accordingly

$$\sigma^2_{\mu_2} = \frac{4}{m^2}S(\rho_s - \bar{\rho})^2\sigma^2_{r_s} + \frac{8}{m^2}S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{\delta r_s \delta r_t\};$$

thus

$$\sigma^2_{\mu_2} = \frac{4}{mn}\left[\frac{S(\rho_s - \bar{\rho})^2(1-\rho_s^2)^2}{m} + \frac{2S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{n\delta r_s \delta r_t\}}{m}\right]$$

$$= \frac{4}{mn}\left[(1-\bar{\rho}^2)^2\mu_2 - 4\bar{\rho}(1-\bar{\rho}^2)\mu_3 + 2(1+\bar{\rho}^2)\mu_4 - 4\bar{\rho}\mu_6 + \frac{2}{m}S(\rho_s - \bar{\rho})(\rho_t - \bar{\rho})\{n\delta r_s \delta r_t\}\right] \quad \text{.....(xxxiv).}$$

The summation term is one of a type which we have seen reasons for supposing very small when it occurred in the expression $\{\delta\bar{r}\,\delta\mu_2\}$, and we think it may again be neglected here. Also the term in $\mu_6$ will be small, since it depends on the sixth powers of the differences of correlation coefficients, and, perhaps, will be adequately accounted for by putting $\beta_4 = 15$, the normal value, i.e. $\mu_6 = 15\mu_2^3$†, or more accurately by taking $\beta_4$ from Table XLII (*b*) on p. 78 of the *Tables for Statisticians and Biometricians*. We have then

$$\sigma^2_{\mu_2} = \frac{4}{mn}[(1-\bar{\rho}^2)^2\mu_2 - 4\bar{\rho}(1-\bar{\rho}^2)\mu_3 + 2(1+\bar{\rho}^2)\mu_4 - 4\bar{\rho}\beta_4\mu_2^3] \quad \text{.....(xxxiv}\textit{bis}\text{),}$$

* Not a bad approximation to $\sigma_{\bar{r}}^2$ may be made as follows:

$$\sigma_{\bar{r}}^2 = \frac{1}{m}\left(\frac{S(\sigma^2_{r_s})}{m} + (m-1)\{\sigma_{r_s}\sigma_{r_t}R_{r_s r_t}\}\right).$$

Now put all the correlations in $\sigma_{r_s}$ and $\sigma_{r_t}$, etc. equal to $\bar{\rho}$ and give $R_{r_s r_t}$ its mean value, i.e.

$$\text{Mean } R_{r_s r_t} = \frac{2}{p+1}\frac{\bar{\rho}(2+p\bar{\rho})}{(1+\bar{\rho})^2}.$$

Then

$$\sigma_{\bar{r}}^2 = \frac{(1-\bar{\rho}^2)^2}{mn}(1+(m-1)\text{ Mean } R_{r_s r_t}).$$

For example, in the Holzinger data Mean $R_{r_s r_t} = \cdot 196{,}432$,

and we find $\sigma_{\bar{r}}^2 = \cdot 0033{,}8074$, not so far removed from the fuller value of (xxxiii), i.e. $\cdot 0033{,}7072$, showing that the terms $-2\mu_2(1-3\bar{\rho}^2) + 4\bar{\rho}\mu_3 + \mu_4$ contribute little to the result.

† $\beta_2 = 2\cdot 0561$ for Holzinger's data, and $= 2\cdot 3104$ for Simpson's data, $\beta_1$ being $\cdot 0003$ and $\cdot 0715$ respectively. Hence $\beta_4$ is nearer 5 in the first case and 7·8 in the second than 15.

a formula open to correction should anyone evaluate successfully the summation term. We conclude therefore that if $\{\delta\bar{r}\,\delta\mu_2\}$, $\sigma_{\bar{r}}^2$ and $\sigma_{\mu_2}^2$ be computed from (xxx), (xxxi) and (xxxiv) respectively and substituted in (xxvii) we shall have the best value at present available of $\sigma_{\sigma_T^2}$, or in more popular and incorrect parlance we shall be able to find the probable error of $\sigma_T^2$, i.e. of the mean square tetrad, and so judge to some extent how far the deviation between its observed and theoretical values is or is not significant.

We have seen that in obtaining the conditions for testing $\sigma_T$ as well as determining the value of $\sigma_T$ itself we have neglected terms of the order $1/(n\sqrt{n})$ as compared with $1/n$. It is accordingly essential that in a really valuable test of the two-factor hypothesis $n$ should be large. This is hardly the case in the two chief series on which the hypothesis has so far been examined, i.e. the data of Holzinger, where $n = 50$, and those of Simpson where $n = 37$ *. Further, our formulae indicate that the size of $m$, the number of correlations, ought also to be considerable†. The number of correlations seems adequate in Simpson's data (91), less so in Holzinger's (36). This number depends on the number of mental abilities measured and very rapidly increases with that. At the same time, of course, the number of tetrads runs up to very unmanageable numbers, but that evil cannot be avoided.

We suggest that some 12 to 15 abilities (66 to 105 correlations, 1485 to 4095 tetrads), the abilities being settled by psychologists *a priori* to avoid "overlaps," are essential to a satisfactory test, the observations to be made on a homogeneous population of several hundreds. Short series involve such large probable errors that a mere statement that theory and observation are in accordance within the limits of the probable errors can carry no conviction with it.

(7) *Restatement of Problem.*

We are not underrating the importance of the problem‡, but really appreciating it, when we state that it is worthy a thorough and direct test of this nature, in which mathematicians and psychologists must work to a common end. The labour involved is great, but not so great that it could not be surmounted by co-operative action.

We should like to word the problem somewhat differently to the Holzinger-Spearman approach to it. Consider the quantity

$$\psi_r^2 = S(r_{su}r_{tv} - r_{tu}r_{sv})^2 \frac{8(m-4)!}{m!}.$$

$\psi_r^2$ is the mean square tetrad and is supposed to be zero for the sampled population; it will only be zero in the sample if every tetrad vanishes. But it will not be zero in

* Simpson's material appears to have been also somewhat heterogeneous, his 37 tested individuals consisting partly of academic and partly of commercial adults of various ages.

† $m$ must be considerable to keep low the two last terms in (xxv) which cannot otherwise be neglected.

‡ The factor problem has bearings not only on mental ability, but on craniometry, where in the cranial correlations there is some reason to believe that a limited number of factors may be concerned.

general for the sample owing to the fact that it consists of a number of squares and therefore is essentially positive, even when the $r$'s only differ from the $\rho$'s by the variations of random sampling. Our first criterion therefore is that $\psi_r{}^2$ should be zero within the limits of random sampling. If it is not, then the sample is not compatible with the sampled population, being of a two-factor constitution. If it is within the limits of a random sample, then the sampled population *may be* of a two-factor constitution; it does not *prove* that it is, it only proves compatibility. There can exist no possible proof that because $(r_{su}r_{tv} - r_{tu}r_{sv})^2$ does not exceed zero by more than, say, twice its probable error, that $\rho_{su}\rho_{tv} - \rho_{tu}\rho_{sv} = 0$, and a two-factor constitution holds for the four variables $s, u, t, v$ in the population sampled. All we can venture to affirm is that it is compatible with it. In the same way $\psi_r{}^2$ lying within reasonable limits of zero judged by its probable error can never prove a general and specific factor constitution, it can only indicate compatibility with such an hypothesis. It is thus easier to show the improbability of a two-factor hypothesis than to demonstrate its existence*.

Let us take our material in triplets of tetrads, and consider first the value for a single tetrad given in (v), i.e.

$$_rT_{st|uv} = (_{\bar{r}}T_{st|uv}) + (\bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv}) + (\delta r_{su}\delta r_{tv} - \delta r_{tu}\delta r_{sv}).$$

We will term the three bracketed expressions $A'$, $B'$ and $C'$ respectively. Now squaring:

$$(_rT_{st|uv})^2 = A'^2 + 2A'B' + 2A'C' + B'^2 + 2B'C' + C'^2,$$

or taking mean values:

$$\{(_rT_{st|uv})^2\} = A'^2 + 2A'\{B'\} + 2A'\{C'\} + \{B'^2\} + 2\{B'C'\} + \{C'^2\}.$$

Now $$A' = {}_{\bar{r}}T_{st|uv} = \frac{\rho_{su}\rho_{tv}(\rho^2_{su} + \rho^2_{tv}) - \rho_{tu}\rho_{sv}(\rho^2_{tu} + \rho^2_{sv})}{2n}.$$

Therefore $A'^2$ is of the order $1/n^2$. $\{B'\} = 0$. $C'$ is of the order $1/n$ and accordingly $A'\{C'\}$ is of the order $1/n^2$. $\{B'^2\}$ is of the order $1/n$, $\{B'C'\}$ and $\{C'^2\}$ of the order $1/n^2$. Hence to a first approximation:

$$\text{Mean (tetrad)}^2 = \{B'^2\}.$$

But $\{B'^2\}$ = mean value of $(\bar{r}_{su}\delta r_{tv} + \bar{r}_{tv}\delta r_{su} - \bar{r}_{sv}\delta r_{tu} - \bar{r}_{tu}\delta r_{sv})^2$, i.e. precisely (as we might anticipate) what we have evaluated in (xiii) and (xiv). We have for mean variation of the triplet of tetrads squared:

$$\{S(_rT)^2\} = \{(r_{su}r_{tv} - r_{tu}r_{sv})^2 + (r_{su}r_{tv} - r_{uv}r_{st})^2 + (r_{st}r_{uv} - r_{tu}r_{sv})^2\}$$
$$= (1/n)(2S(\rho^2) - 6P_3 + 12\lambda^2) \quad \text{.....(xxxv),}$$

* Because a tetrad vanishes within the limits of its probable error we cannot assert, as some authorities have done, that it is clearly demonstrated that what is common to $s$, $t$ and $u$ is also shared by $v$. Such writers forget that we can increase the probable error to any extent by reducing the size, $n$, of the sample and so can demonstrate for $n = 30$ or 50 what would be most improbable for $n = 500$ or 1000.

where $P_3$ = third order products and $\lambda = \rho_{su}\rho_{tv} = \rho_{tu}\rho_{sv} = \rho_{st}\rho_{uv}$. Now, if

$$\Delta_{stuv} = \begin{vmatrix} 1, & \rho_{st}, & \rho_{su}, & \rho_{sv} \\ \rho_{ts}, & 1, & \rho_{tu}, & \rho_{tv} \\ \rho_{us}, & \rho_{ut}, & 1, & \rho_{uv} \\ \rho_{vs}, & \rho_{vt}, & \rho_{vu}, & 1 \end{vmatrix},$$

we have on expansion $\Delta_{stuv} = 1 - S(\rho^2) + 2P_3 - 3\lambda^2$, and can get rid of $P_3$. Accordingly

$$\{S({}_rT)^2\} = (1/n)(s - 3\Delta_{stuv} - S(\rho^2) + 3\lambda^2)$$
$$= (1/n)((1-\rho^2_{su})(1-\rho^2_{tv}) + (1-\rho^2_{sv})(1-\rho^2_{tu}) + (1-\rho^2_{st})(1-\rho^2_{uv})) - 3\Delta_{stuv}.$$

Hence if $D$ be the general determinant of the correlation system, i.e. $|\rho|$, and we denote by $D_{(abcd...)}$ the minor obtained by striking out all constituents not in both the *abcd...* rows and columns, we shall have

$$\{\psi_r{}^2\} = \frac{8\times(m-4)!}{m!}\,\frac{\Sigma\,(D_{(su)}D_{(tv)} + D_{(sv)}D_{(tu)} + D_{(st)}D_{(uv)} - 3D_{(stuv)})}{n} \qquad \text{.....(xxxvi).}$$

The summation $\Sigma$ refers to every quadruplet of four variables out of the total $m$*.

We now need to find the standard deviation of $\psi_r{}^2$.

Let us write $\xi = ({}_rT_{st|uv})^2 = (A' + B' + C')^2$, then what we need is $\{S(\xi^2)\} - \{\psi_r{}^2\}^2$. But $\xi^2 = (A' + B' + C')^4$ and $\{\xi^2\} = \{B'^4\}$ + terms of orders $1/n^3$ and $1/n^4$, while $\{B'^4\}$ is of order $1/n^2$. But to discover the value of $\{B'^4\}$ we want to know all the fourth order products of the variations in the correlation coefficients. These are at present unknown, and having regard to the known values of $\beta_1$ and $\beta_2$ for distributions of $r$ (see Tables, *Biometrika*, Vol. XI. pp. 379 *et seq.*) it seems rash to assume normality. Actually each tetrad is supposed to be varying round zero with its own standard deviation due to random sampling. We have as yet no proof that this variation takes the form of a normal curve. And if any single correlation coefficient does not, what chance is there that the tetrad $r_{su}r_{tv} - r_{sv}r_{tu}$, based upon four inter-correlated non-normally distributed variates, is going to do so? If we assume that it does so roughly and approximately, what evidence have we that a distribution based upon the variations of a large number of tetrads, *each with its own standard deviation*†, will follow a *normal curve* the standard deviation of which is based upon an approximation to the variance of the mean sum of the squared tetrads? If the deviation of each tetrad from zero had been divided by its own standard deviation we should have more hope of the resulting ratios obeying a normal law. It does not seem accordingly to be any demonstration of the validity of the theory of a general and specific factor to place a normal curve on the top of a frequency distribution of tetrads and argue from its appearance to the truth of the hypothesis. No proof

* We have given this form in the hope that some abler mathematician might possibly throw back the summation $\Sigma$ of minors into a simple determinantal form.

† A very little consideration shows how diverse are the variances of tetrads. If the six correlations concerned in the tetrad are all equal the standard deviation is $0/\sqrt{n}$, $\cdot375/\sqrt{n}$, $\cdot5/\sqrt{n}$, $\cdot375/\sqrt{n}$ and $0/\sqrt{n}$ as the correlations take the value 0, ·25, ·5, ·75 and 1 in succession.

has yet been provided that the distribution ought to follow a normal curve at all, and if such a proof can be provided, then the degree of accuracy of fit must be properly tested. It is quite conceivable also that 378 or 1000 tetrads arising from a number of correlation coefficients with random sampling values might when "tetraded" give a distribution very akin to a normal curve. We do not say that they would or would not, but it is quite clear that confident proofs of an hypothesis cannot be based on a curve which it has not been demonstrated that the distribution ought really to satisfy. Writers on this subject do not tell us whether they suppose single tetrads to follow the normal law, nor how if they do a number of tetrads not measured in terms of their own standard deviations will also follow a normal law, and this with a standard deviation which is not even a true mean of the standard deviations of the contributions. Whether a theory be true or false can only be ascertained when the theory has been adequately developed so that it can be tested against observed facts. Here the facts are placed against a result which has not been shown to flow from the theory—and the accordance then is said to be good!

(8) *Fourth Moment of Tetrad Distribution.*

If notwithstanding the warning given as to the nature of the distributions of $r_{st}$ and $r_{uv}$, we assume that as a first approximation we may treat them as normally distributed, then if $r_1$, $r_2$, $r_3$, $r_4$ be any four correlation coefficients and $R_{12}$, $R_{13}$, $R_{14}$, $R_{23}$, $R_{24}$, $R_{34}$ their correlations as determined by the Pearson-Filon formulae, we require the following mean products:

$$\{\delta r_1^4\} = \frac{3}{n^2}(1-r_1^2)^4, \quad \{\delta r_1^3 \delta r_2\} = \frac{3}{n^2} R_{12}(1-r_1^2)^3(1-r_2^2),$$

$$\{\delta r_1^2 \delta r_2 \delta r_3\} = \frac{1}{n^2}(2R_{12}R_{13}+R_{23})(1-r_1^2)^2(1-r_2^2)(1-r_3^2),$$

$$\{\delta r_1^2 \delta r_2^2\} = \frac{1}{n^2}(1+2R_{12}^2)(1-r_1^2)^2(1-r_2^2)^2,$$

$$\{\delta r_1 \delta r_2 \delta r_3 \delta r_4\} = \frac{1}{n^2}(1-r_1^2)(1-r_2^2)(1-r_3^2)(1-r_4^2)(R_{12}R_{34}+R_{13}R_{24}+R_{14}R_{23})$$

.....(xxxvii),

where it is clear that the last contains implicitly the first four*. It is accordingly a problem whether the last is reducible to any more manageable form. Up to the present we have not succeeded in finding it.

If we write

$$a = r_{13}\delta r_{24}, \quad b = r_{24}\delta r_{13}, \quad c = r_{14}\delta r_{23}, \quad d = r_{23}\delta r_{14}, \quad e = r_{12}\delta r_{34}, \quad f = r_{34}\delta r_{12},$$

then the fourth moment coefficient of the triplet of tetrad differences for the variables 1, 2, 3, 4 is given by

$$3M_4' = \{(B')^4\} = \{(a+b-c-d)^4 + (e+f-a-b)^4 + (c+d-e-f)^4\},$$

* For proof of the last see Soper, *Biometrika*, Vol. IX. 1913, p. 100, and Isserlis, Vol. XI. 1916, p. 189.

or, expanding and rearranging,

$$\begin{aligned}3M_4' = \{&2(a^4+b^4+c^4+d^4+e^4+f^4) - 4a^3(-2b+c+d+e+f)\\ &- 4b^3(-2a+c+d+e+f)\\ &- 4c^3(a+b-2d+e+f) - 4d^3(a+b-2c+e+f) - 4e^3(+a+b+c+d-2f)\\ &- 4f^3(a+b+c+d-2e) + 12(a^2b^2+c^2d^2+e^2f^2)\\ &+ 6(a^2c^2+a^2d^2+a^2e^2+a^2f^2+b^2c^2+b^2d^2+b^2e^2+b^2f^2+c^2e^2+c^2f^2+d^2e^2+d^2f^2)\\ &+ 12a^2(cd+ef-bc-bd-be-bf) + 12b^2(ef+cd-ac-ad-ae-af)\\ &+ 12c^2(ab+ef-ad-bd-ed-fd) + 12d^2(ab+ef-ac-bc-ec-fc)\\ &+ 12e^2(ab+cd-fa-fb-fc-fd) + 12f^2(ab+cd-ae-be-ce-de)\\ &+ 24(abcd+abef+cdef)\} \quad \ldots\ldots\ldots\

culating the value of $M_4'$ even for a single triplet of tetrads is alarming enough, for a series of 1000 triplets of tetrads actually appalling! Nor are we certain that the great labour would be worth while, for the result is based on hypotheses which we know are inaccurate, and which do not lead to even reasonable approximations when the correlations are fairly large and the value of $n$ small. To obtain, however, a rough estimate of the value of $M_4'$, let us assume $r_{12} = r_{13} = r_{14} = r_{23} = r_{24} = r_{34} = \bar{\rho}$ (as we have done as a first approximation for the case of $M_2$). We find

$$\left.\begin{aligned} R_{24|13} = R_{21|43} = R_{23|14} &= \frac{2\bar{\rho}^2}{(1+\bar{\rho})^2} \\ R_{12|13} = R_{14|13} = R_{12|14} = \text{etc.} &= \frac{\bar{\rho}\,(2+3\bar{\rho})}{2\,(1+\bar{\rho})^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xl)},$$

whence on substituting we have

$$3n^2 M_4' = 36\bar{\rho}^4(1-\bar{\rho}^2)^4\left(4 + \frac{16\bar{\rho}^2}{(1+\bar{\rho})^2} + \frac{16\bar{\rho}^4}{(1+\bar{\rho})^4} - \frac{8\bar{\rho}\,(2+3\bar{\rho})}{(1+\bar{\rho})^2} + \frac{4\bar{\rho}^2(2+3\bar{\rho})^2}{(1+\bar{\rho})^4} - \frac{16\bar{\rho}^3(2+3\bar{\rho})}{(1+\bar{\rho})^4}\right),$$

and finally

$$M_4' = \frac{48\bar{\rho}^4(1-\bar{\rho})^4}{n^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xli)},$$

a very simple form. Now on the same hypothesis (see the first term of (xxv)) the value of $M_2'$ is given by

$$M_2' = \frac{4\bar{\rho}^2(1-\bar{\rho})^2}{n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xlii)}.$$

Here for the simple tetrads* treated as a symmetrical distribution $M_2'$ and $M_4'$ would be the second and fourth moment coefficients. For the squared tetrads $M_2'$ would be the mean and

$$\Sigma^2 = M_4' - M_2'^2 = \frac{32\bar{\rho}^4(1-\bar{\rho})^4}{n^2}$$

the squared standard deviation, i.e.

$$\Sigma = \frac{4\sqrt{2}}{n}\,\bar{\rho}^2(1-\bar{\rho})^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xliii)}.$$

Turning to the distribution of tetrads rendered symmetrical, we have of course $\beta_1 = 0$. Further $\beta_2 = M_4'/M_2'^2 = 3$, or we conclude that these conditions for normality as far as $\beta_1$ and $\beta_2$ are concerned would be satisfied if all the correlations had the same equal value, but in this case the standard deviations of all the individual tetrads due to random sampling become the same and there is some theoretical reason for pooling the tetrads which are based on correlations all selected from the same curve. This curve is known to be

$$y_n = \frac{(1-\rho^2)^{\frac{1}{2}(n-1)}}{\pi\,(n-3)!}(1-r^2)^{\frac{1}{2}(n-4)}\frac{d^{n-2}}{d\,(r\rho)^{n-2}}\left(\frac{\cos^{-1}(-\rho r)}{\sqrt{1-\rho^2 r^2}}\right),$$

of which the $\beta_2$ is *not* however 3.

* Of course if we treat all correlations as the same these values are true for the whole series of tetrads as well as for a triplet of three tetrads.

Why then does the distribution of the tetrads give $\beta_2 = 3$ in the present case? For the simple reason that the Pearson-Filon formulae (viii) and the formulae (xxxvii) are all based on the assumption that the distribution of $r$ may to a first approximation—and *Biometrika*, Vol. XI. pp. 379 *et seq.* show it to be a *very* rough approximation—be treated as normal! We have dealt with this point at length because we wished to indicate that it is a fallacious theory and not the essential nature of the two-factor hypothesis which leads to the above suggestion of approximate normality. We do not think it at all possible, in the present state of our theoretical knowledge, to deduce any evidence in favour of the two-factor hypothesis from coincidence or non-coincidence of the distribution of tetrads with a normal curve. We shall see as we advance that such a coincidence is as observationally improbable as it is theoretically unreasonable.

We might learn something of the real nature of the distribution of tetrads were $M_4'$ in (xxxix) evaluated as $M_2'$ in (xxvi) at least as far as the terms in $\sigma_r^2$ and its powers. Until this not very promising task is accomplished, we must content ourselves with comparing (xli) and (xlii) with the results obtained from observations*. We shall not, however, be able to ascertain whether the divergences found are due to (i) neglect of terms or (ii) to failure of the two-factor hypothesis in our material or to a combination of both!

(9) *Results which flow if we assume the Tetrad Distribution to be Normal.*

In the present state of our knowledge it seems that we can only apply a *single* test to determine whether $\{\psi_r^2\}$ is zero within the limits of random sampling; we can compare its theoretical value computed by (xvi), or if they are found to be close enough by the short formulae (xxv) or (xxvi) with its observed value, and determine whether the difference is significant having regard to the approximate value of $\sigma_{\psi_r^2}$ or $\sigma_{\sigma_T^2}$ provided by (xxvii). But here we must remember that (*a*) not knowing the theoretical distribution of the tetrads we shall be judging of the identity of a theoretical and an observed distribution on the basis of a single constant, which is extremely inadequate, and (*b*) we shall be judging by a "probable error" which fails more and more in any meaning as the distribution of tetrads diverges more and more from normality. Professor Spearman appears to get over these difficulties (*a*) and (*b*) by assuming without any discussion or show of proof that the distribution of tetrads must follow a normal curve. We know that correlation coefficients only roughly approximate to a normal distribution; it is inconceivable therefore, even if the original variates follow normal distributions, that the tetrads should do so. Thus it seems worth while, since half the strength at least of the Spearman demonstration appears to turn on the distribution of tetrads being normal, to investigate a little more closely what must hold if the tetrad distribution were really a normal one. We can do this best by inquiring into what would be

* For Holzinger's data the observed standard deviation of the squared tetrads is $\Sigma = \cdot 007{,}465$, while the value obtained from (xliii) is $\Sigma = \cdot 005{,}821$. For Simpson's data the corresponding values are $\cdot 018{,}002$ and $\cdot 009{,}542$! It will therefore be clear that but little is to be learnt of the success or failure of the two-factor hypothesis from $M_4'$ until we know its value far more closely.

the distribution of squared tetrads if we suppose the distribution of tetrads themselves to follow the normal curve

$$y = y_0 e^{-\frac{1}{2}\frac{T^2}{\sigma_T^2}},$$

where $T$ is a tetrad measured from zero and $\sigma_T$ is the standard deviation of tetrads.

Let $u$ be a squared tetrad $= T^2$, then $2TdT = du$ or $dT = \frac{1}{2}u^{-\frac{1}{2}}du$, and consequently the frequency distribution of squared tetrads is given by

$$z = z_0 e^{-\frac{1}{2}\frac{u}{\sigma_T^2}} u^{-\frac{1}{2}}.$$

This is a curve of Pearson's Type III or

$$z = z_0 e^{-\gamma u} u^p \quad \text{.....(xliv),}$$

where $p = -\frac{1}{2}$ and $\gamma = 1/(2\sigma_T^2)$.

Thus we have:

$$\text{Mean squared tetrad,} \quad \mu_1' = \frac{p+1}{\gamma} = \sigma_T^2 \quad \text{.....(xlv),}$$

as might of course be foreseen.

If $\Sigma =$ the standard deviation of squared tetrads,

$$\Sigma^2 = \frac{p+1}{\gamma^2} = 2\sigma_T^4,$$

or

$$\Sigma = \sqrt{2}\sigma_T^2 \quad \text{..... .....(xlvi).}$$

Further we have for the fundamental constants $\beta_1$ and $\beta_2$ of the distribution

$$\beta_1 = \frac{4}{p+1} = 8, \quad \beta_2 = \frac{3(p+3)}{p+1} = 15 \text{.....(xlvii).}$$

Apart from the general test of "goodness of fit" of the tetrad distribution to a normal curve, which we shall deal with later, a very simple test arises from the quartile of the normal curve. The quartile will be the median tetrad if we disregard sign. But for the normal curve the quartile is $\cdot 674{,}490\sigma_T$. Thus we should have

$$\text{Median tetrad} = \cdot 674{,}490\sigma_T \quad \text{.....(xlviii),}$$

and the probable error of this median tetrad will be $(\cdot 674{,}490)^2 \sigma_{\sigma_T}$, which can be determined from our knowledge of $\sigma^2_{\sigma_T^2}$ in (xxvii) with reasonable approximation.

Again if the distribution be normal, the mean tetrad $\overline{T}$, all tetrads being treated as positive, is given by

$$\overline{T} = \sqrt{\frac{2}{\pi}}\sigma_T = \cdot 797{,}884\sigma_T \quad \text{.....(xlix),}$$

and the probable error of $\overline{T}$ is given by

$$\cdot 674{,}490\sigma_{\overline{T}} = \cdot 538{,}165\sigma_{\sigma_T}.$$

The tests involved in (xlviii) and (xlix) are easy to apply when $\sigma_{\sigma_T^2}$ has been computed, for very approximately

$$\sigma_{\sigma_T} = \frac{1}{2\sigma_T} \cdot \sigma_{\sigma_T^2}.$$

With only the single test such as the Spearman process provides one is apt to be suspicious of that test being easily satisfied by some very simple hypothesis other than an elaborate structural one. We might even devise one which gave practically the desired normal distribution of the tetrads. Let us suppose the correlations on which the tetrads are based were drawn *at random* from a symmetrical frequency distribution of mean $\rho$ and standard deviation $\sigma_\rho$. In this case there would be no correlation between the correlations, or if $\rho + \delta r_s$, $\rho + \delta r_t$ be two correlations drawn at random, then $\{\delta r_s \delta r_t\} = 0$.

Taking any four correlations $\rho + \delta r_1$, $\rho + \delta r_2$, $\rho + \delta r_3$ and $\rho + \delta r_4$ we have:

$$\begin{aligned} T &= (\rho + \delta r_1)(\rho + \delta r_2) - (\rho + \delta r_3)(\rho + \delta r_4) \\ &= \rho(\delta r_1 + \delta r_2 - \delta r_3 - \delta r_4) + \delta r_1 \delta r_2 - \delta r_3 \delta r_4; \end{aligned}$$

taking the mean of both sides for all $r$'s:

$$\{T\} = \rho(\{\delta r_1\} + \{\delta r_2\} - \{\delta r_3\} - \{\delta r_4\}) + \{\delta r_1 \delta r_2\} - \{\delta r_3 \delta r_4\},$$

but all these curled bracket terms vanish. Hence the mean tetrad is zero. Again

$$\begin{aligned} T^2 = \rho^2 [&(\delta r_1)^2 + (\delta r_2)^2 + (\delta r_3)^2 + (\delta r_4)^2 + 2(\delta r_1 \delta r_2) \\ &- 2(\delta r_1 \delta r_3) - 2(\delta r_1 \delta r_4) - 2(\delta r_2 \delta r_3) - 2(\delta r_2 \delta r_4) + 2(\delta r_3 \delta r_4)] \\ &+ 2\rho(\delta r_1 + \delta r_2 - \delta r_3 - \delta r_4)(\delta r_1 \delta r_2 - \delta r_3 \delta r_4) \\ &+ (\delta r_1 \delta r_2 - \delta r_3 \delta r_4)^2. \end{aligned}$$

Taking mean values:

$$\text{Mean } T^2 = \sigma_T{}^2 = \rho^2 4\sigma_\rho{}^2 + 2\rho \times 0 + 2\sigma_\rho{}^2 \sigma_\rho{}^2,$$

or

$$\sigma_T{}^2 = 2\sigma_\rho{}^2 (2\rho^2 + \sigma_\rho{}^2) \quad \ldots\ldots\ldots\

Thus far we have not assumed that the symmetrical distribution from which the $r$'s are drawn is normal. If we now suppose it to be normal, then $\nu_4 = 3\nu_2^2 = 3\sigma_\rho^4$ and

$$\{T^4\} = 3\,(2\sigma_\rho^2\,(2\rho^2 + \sigma_\rho^2))^2 + 12\sigma_\rho^8 \quad \text{.....(lii).}$$

Hence we have for the fundamental constant

$$B_2 = \{T^4\}/(\{T^2\})^2 = 3\left(1 + \left(\frac{\sigma_\rho^2}{2\rho^2 + \sigma_\rho^2}\right)^2\right) \quad \text{.....(liii).}$$

The system of tetrads deduced from a normal distribution of correlation coefficients is theoretically a leptokurtic curve, but it would be in appearance extremely like a normal curve with its tails in reality slightly emphasised. Indeed the resemblance would be very great if $\rho$ and $\sigma_\rho$ were to be so chosen that the chance of a randomly selected correlation coefficient being negative was very improbable. Thus*:

If $\rho = 2\sigma_\rho$ we should have $B_2 = 3{\cdot}037$,
$\rho = 2{\cdot}5\sigma_\rho$ „ „ „ $B_2 = 3{\cdot}016$,
$\rho = 3\sigma_\rho$ „ „ „ $B_2 = 3{\cdot}008$.

It may be safely asserted that even the most highly trained statistical judgment would not detect by an ocular examination of graphs such deviations from normality as are represented by these values of $B_2$. It may be reasonably doubted if it could discriminate in the same manner between $B_2 = 3{\cdot}0$ and $B_2 = 3{\cdot}1$.

We can easily measure the amount of variation there will be in random samples of $M$ correlation coefficients from the distribution $\rho$, $\sigma_\rho$, for since

$$\sigma^2_{T_r} = 2\sigma_r^2\,(2\bar{r}^2 + \sigma_r^2),$$

where $\bar{r}$ is the mean $r$ of samples as distinguished from the true $\rho$, it follows that

$$\sigma_T \delta\sigma_T = 4\,(\sigma_\rho(\rho^2 + \sigma_\rho^2)\,\delta\sigma_r + \sigma_r^2\rho\delta\bar{r}).$$

Hence squaring, taking the means and substituting the values

$$\{(\delta\sigma_r)^2\} = \sigma_\rho^2/(2M), \quad \{(\delta\bar{r})^2\} = \sigma_\rho^2/M$$

and $\{\delta\bar{r}\,\delta\sigma_r\} = 0$—for in the case of material following the normal curve there is no correlation between the mean and the standard deviation of samples—we have:

$$\sigma_{\sigma_T} = \frac{4\sigma_\rho^2}{\sqrt{M}}\sqrt{\frac{\frac{1}{2}(\rho^2 + \sigma_\rho^2)^2 + \rho^2\sigma_\rho^2}{2\sigma_\rho^2(2\bar{\rho}^2 + \sigma_\rho^2)}} \quad \text{.....(liv).}$$

Now let us consider the following system and compare it with Holzinger's data. Suppose we had a normal curve of mean $\rho = {\cdot}340{,}000$ and of standard deviation $\sigma_\rho = {\cdot}111{,}000$, and let us suppose 36 correlations selected from this distribution at random, then the mean $r$ of the samples would vary round $\rho$ in the manner prescribed by

$${\cdot}340{,}000 \pm {\cdot}012{,}478,$$

and the standard deviations of these samples in the manner prescribed approximately by

$${\cdot}111{,}000 \pm {\cdot}008{,}823.$$

* For the $\rho$ and $\sigma_\rho$ of Holzinger's data $\rho = 2{\cdot}884\sigma_\rho$, while for Simpson's data $\rho = 2{\cdot}333\sigma_\rho$.

The tetrads would follow a very nearly normal curve round a zero tetrad having its

$$B_2 = 3{\cdot}0077,$$

and so be quite indistinguishable from a normal curve of errors.

Further the standard deviation (see (l)) of this distribution and its probable error would be

$$\cdot 077{,}465 \pm \cdot 010{,}386.$$

Now in Holzinger's data the mean of the correlations is ·347,778 and their standard deviation ·120,603. The first is only 0·62 times the probable error in excess of $\rho$, and the second is only 1·1 times the probable error in excess. Holzinger's data have a moderate resemblance to a normal curve distribution as a graph, and finally their $\sigma_T$ is equal to ·077,253 by observation, or the deviation from ·077,465 is only about ·02 times the probable error. Reduced to probable errors the expected and observed values are both ·052, when stated only to two significant figures!

Now if to show a general resemblance by a graph to a normal curve and to state that the observed and expected values are ·051 and ·052* suffices to demonstrate the hypothesis of two factors on Holzinger's data, will not a similar reasoning enable us to assert that Holzinger's tetrad curve simply results from forming the tetrads of a random selection of 36 correlations drawn from a normal distribution of correlation coefficients with a mean and standard deviation well within the probable error limits of the like quantities for Holzinger's data?

Now we by no means suggest that drawings from a normal curve were the origin of Holzinger's correlations of mental abilities; we know they were not obtained in this way. But we think it reasonable to suggest that the one-constant proof adopted to demonstrate a delicate structural constitution for these correlations would equally well demonstrate that they arose merely from random samplings of a normal curve. The fact is that 36 correlation coefficients are not in such a case adequate to discriminate between random drawings of the correlations and an elaborate structural hypothesis as to the nature of the variates. The distribution of correlation coefficients is not generally even approximately normal. It may be seen at once that we could not apply the same reasoning to Simpson's data. Here $\bar{r} = \cdot 486{,}703$ and $\sigma_r = \cdot 208{,}617$, and $\sigma_r$ is relatively too great for us to assume that the correlations are selected from a normal curve. Indeed an inspection of the following coarsely grouped frequency distribution of the 91 correlation coefficients indicates that it can hardly be looked upon as a sample from a normal

*Central Values of Coefficients.*

| | *·945* | *·845* | *·745* | *·645* | *·545* | *·445* | *·345* | *·245* | *·145* | *·045* |
|---|---|---|---|---|---|---|---|---|---|---|
| Frequency ... | 3 | 5 | 10 | 6 | 21 | 11 | 12 | 18 | 3 | 2 |

* Values given by Spearman, *The Abilities of Man*, p. 149, with the remark "Evidently the agreement could hardly be bettered."

distribution; we appear to be dealing with a bimodal if not a trimodal system, and the sample would be almost as probable from a rectangle ($\rho = \cdot 500{,}000$, $\sigma_\rho = \cdot 288{,}675$) as from a normal curve. But this very irregularity of the correlation coefficients, which may be quite well inherent in the mental abilities selected for testing, makes it *a priori* exceedingly improbable that any frequency distribution based on these correlations by forming their tetrads will give any approach to a normal curve. In the case of Holzinger's data there is likely to be more resemblance to a normal curve because his correlations are more nearly a sample from a normal distribution.

(10) *Tests to be applied to Data.*

We will now turn to a consideration of the data which are more or less sufficiently ample to allow of legitimate testing of the hypothesis of general and specific factors. These data practically reduce to two series, that of Holzinger and that of Simpson, although even here some criticism may be made of their complete suitability. In both series the number of persons tested is too small, and in Simpson's series it is possibly too heterogeneous. However it is on these series that the greatest stress has been laid and they have been spoken of as demonstrating with the accuracy of physical investigation the truth of the two-factor hypothesis. The demonstration consists in asserting (i) that the tetrad distributions are closely normal and (ii) that these normal distributions have a probable error value closely identical with what is theoretically to be expected, i.e. a value deducible from Equation (xxvi).

To test these demonstrations we have first formed the distribution of the tetrads for both series, we have then found the distribution of the squared tetrads*. We have finally found by brute force the value of $\sigma_T^2$ from Equation (xvi) to test the efficiency of Formula (xxvi).

(11) *Illustration I. Holzinger's Data.*

The following table is given by Spearman and Holzinger†. It is said to "describe an experiment based on 50 cases and comprising an unusually homo-

TABLE I.

*Correlations between Tests.*

| Tests | *a* | *b* | *c* | *d* | *e* | *f* | *g* | *h* | *i* |
|---|---|---|---|---|---|---|---|---|---|
| *a* | — | ·50 | ·54 | ·34 | ·47 | ·40 | ·50 | ·33 | ·24 |
| *b* | ·50 | — | ·39 | ·56 | ·51 | ·43 | ·36 | ·32 | ·15 |
| *c* | ·54 | ·39 | — | ·49 | ·52 | ·27 | ·27 | ·26 | ·27 |
| *d* | ·34 | ·56 | ·49 | — | ·30 | ·27 | ·52 | ·13 | ·35 |
| *e* | ·47 | ·51 | ·52 | ·30 | — | ·35 | ·14 | ·33 | ·18 |
| *f* | ·40 | ·43 | ·27 | ·27 | ·35 | — | ·38 | ·40 | ·19 |
| *g* | ·50 | ·36 | ·27 | ·52 | ·14 | ·38 | — | ·19 | ·38 |
| *h* | ·33 | ·32 | ·26 | ·13 | ·33 | ·40 | ·19 | — | ·29 |
| *i* | ·24 | ·15 | ·27 | ·35 | ·18 | ·19 | ·38 | ·29 | — |

* Not from the distribution of the tetrads, but from squaring each individual tetrad and then grouping.

† *British Journal of Psychology*, Vol. XVI. Part 2, p. 87.

geneous group of pupils aged 12 and 13 years. The nine tests used were the components of the Otis Intelligence Scale, the parts being very similar to the American Army Tests." The data are due to Holzinger.

There are 378 independent tetrads here leading to 126 triplets of tetrads, each triplet depending on a group of four characters, $s$, $t$, $u$, $v$, for which in the sampled population the relations

$$\rho_{su}\rho_{tv} = \rho_{tu}\rho_{sv} = \rho_{st}\rho_{uv}$$

are assumed to hold.

Thus by (xiv) each triplet of tetrads will contribute

$$\begin{aligned} 2/n\,[&(\rho^2_{su} + \rho^2_{st} + \rho^2_{sv} + \rho^2_{tu} + \rho^2_{tv} + \rho^2_{uv}) \\ &- 3\,(\rho_{tu}\rho_{tv}\rho_{uv} + \rho_{su}\rho_{sv}\rho_{uv} + \rho_{st}\rho_{sv}\rho_{vt} + \rho_{st}\rho_{tu}\rho_{su}) \\ &+ 2\,(\rho_{st}\rho_{uv}\rho_{sv}\rho_{tu} + \rho_{sv}\rho_{tu}\rho_{su}\rho_{tv} + \rho_{su}\rho_{tv}\rho_{st}\rho_{uv})] \\ &= 2/n\,[S\,(\rho^2) - 3P_3 - 2P_4], \text{ say } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(lv)}. \end{aligned}$$

Since this paper has been in type Professor Spearman has stated that his book (*The Abilities of Man*) never made any assumption as to the theoretical normality of the tetrad distribution. He seems to overlook what he has himself written. On p. 140 he says: "The most perfect comparison between the two, the observed tetrad differences and those to be expected from sampling errors alone, is obtained by making a complete frequency distribution of each of these two sets of values. A more summary comparison is got by seeing whether or not about half of the observed tetrad differences are greater and half less than their probable error. Much the same thing is to see whether the median observed tetrad difference and the probable error are about equal." In the "most perfect comparison" Professor Spearman invariably puts a normal curve on his diagram as representing the frequency distribution of theory. His "more summary comparisons" are again entirely based on the theory being a normal curve. There is no meaning in the statement that the median tetrad difference should equal the probable error, unless the normal curve has been taken as the theoretical distribution.

Further illustrations are:

p. 142: "The theoretical distribution is given ... in the curve" [normal curve].

p. 143: "The two distributions are shown in the same way as in the preceding case" [normal curve].

p. 146: "The two distributions, curve [normal curve] and rectangles ... display one of the most striking agreements between theory and practice ever recorded in psychology."

p. 149: "The two distributions theoretical [normal curve] and observed.... Evidently the agreement could hardly be bettered."

p. 154: "The two distributions theoretical and observed are shown in the same way as on p. 142."

Again on this p. 154 in two diagrams curves which are presumably intended for normal curves are used, and approximation to the normal curve is treated as a test of the accordance of observation and theory. Further illustration is unnecessary.

With the six correlations of a quadruplet of variates arranged thus:

$$\begin{array}{cc|c} \rho_{su} & \rho_{sv} & \rho_{st} \\ \rho_{tu} & \rho_{tv} & \rho_{uv} \end{array}$$

it was found fairly easy, if laborious, to compute:

(*a*) the value of each tetrad,

(*b*) the square of each tetrad,

(*c*) 21 times the sum of the squares of each correlation coefficient,

(*d*) the sum of the triple products leaving out one variate each time, and

(*e*) the sum of the quadruple products.

Each of the operations (*c*), (*d*), (*e*) was done separately and continuously, so that the attention of the computer might be fixed on a single process at one time.

If $\Sigma$ denotes a summation for the 126 triplets we found:

$$\Sigma(S(\rho^2)) = 102{\cdot}4338 = 21 \times 4{\cdot}8778,$$

$$\Sigma(P_3) = 21{\cdot}751{,}944, \quad \Sigma(P_4) = 5{\cdot}579{,}378.$$

Hence:
$$\{\psi_r{}^2\} = \frac{96{\cdot}673{,}448}{378 \times 50} = {\cdot}005{,}1150 = \sigma_T{}^2.$$

Thus:
$$\sigma_T = {\cdot}071{,}519.$$

The weak point of this result lies of course in the fact that the formula above holds for the $\rho$'s and there is no alternative but replacing them by the observed $r$'s. This involves neglecting terms of the order $1/\sqrt{n}$, i.e. in this case $1/\sqrt{50}$ as compared with unity, that is the terms neglected may be about $\frac{1}{7}$ of those retained. No stronger argument could be used for largely increasing the number of persons tested.

We now proceeded to determine the constants of the distribution of correlation coefficients*. We have:

$$\bar{r} = {\cdot}347{,}778, \qquad \sigma_r = {\cdot}120{,}603,$$

$$\mu_2 = {\cdot}0145{,}4500, \qquad \mu_3 = {\cdot}0000{,}3216, \qquad \mu_4 = {\cdot}0004{,}3499,$$

$$\beta_1 = {\cdot}000{,}336, \qquad \beta_2 = 2{\cdot}056{,}107.$$

Accordingly from (xxv):

$$\sigma_T{}^2 = \tfrac{4}{50}[{\cdot}0514{,}5115 + {\cdot}0151{,}1058 - {\cdot}0000{,}0003 + {\cdot}0000{,}0017],$$

where we indicate the magnitude of each contributory term. Hence:

$$\sigma_T{}^2 = {\cdot}0053{,}2495,$$

* Without grouping by squaring, cubing and fourth powering the correlation coefficients.

or if we omit the $\mu_3$ and $\mu_4$ terms, $\sigma_T^2 = \cdot 0053,2492_5$, thus they only make a difference in the eighth decimal place, and as far as (xxv) goes we should be justified in using (xxvi). We have then $\sigma_T = \cdot 072,972$. But we have seen that $\sigma_T$ as ascertained from the full formula (lv) is $\cdot 071,519$. Thus we see that $\sigma_T$ as found from the approximate formula differs from the exact value *in the second significant figure.* This can only be due to our replacing $[X_1X_2]$ by $\{X_1X_2\}$: see p. 256. But as we shall see the probable error of $\sigma_T$ is itself only significant in the third place of decimals, the very place in which the theoretical value of $\sigma_T$ as given by (xxv) is inexact. It is therefore very doubtful whether we can replace the more exact but very laborious (lv) by the Spearman-Holzinger formula (xxvi), which is a good enough approach to (xxv), for the error involved is of the order of the probable error of $\sigma_T$. The source of the trouble seems to be that Spearman and Holzinger have not recognised that their $\Sigma(X_1X_2)$ is not the same thing as the $S(X_1X_2)$ by which they replace it. We have seen so far no short method of evaluating $[X_1X_2]$ except by forming the 378 products and taking their mean value. We next have to calculate $\sigma^2_{\sigma_T^2}$ from (xxvii), but this involves the determination of $\sigma^2_{\bar{r}^2}$, $\sigma^2_{\mu_2}$ and $\{\delta\bar{r}\,\delta\mu_2\}$. We note that $p = 9$, $m = 36$ and $n = 50$ for Holzinger's data.

We have from (xxxiii): $\sigma^2_{\bar{r}^2} = \cdot 0033,70716,$

and from (xxxiv *bis*): $\sigma^2_{\mu_2} = \cdot 0000,27008$*,

while from (xxx): $\{\delta\bar{r}\,\delta\mu_2\} = -\cdot 0000,19136.$

The necessary coefficients in (xxvii) are

$$\bar{\rho}(1-\bar{\rho})(1-2\bar{\rho}) + \frac{3(1-2\bar{\rho})}{m-1}\mu_2 = \cdot 071589$$

and

$$1 + \frac{6\bar{\rho}(1-\bar{\rho})}{m-1} = 1\cdot 0388,8486.$$

Finally we obtain $\sigma_{\sigma_T^2} = \cdot 0005,7994,$

or, in the language of "probable error,"

Probable Error of $\sigma_T^2 = \cdot 0003,9117.$

We have now to compare these theoretical results, i.e. $\sigma_T^2 = \cdot 005,115 \pm \cdot 000,391$, with the observed value of $\sigma_T^2$.

At first sight it might appear relatively easy to obtain the observed value of $\sigma_T^2$, but this is far from the case. It may be approached by three different paths. It should be:

(*a*) the squared standard deviation of the symmetrical distribution of tetrads,

(*b*) the mean of the distribution of squared tetrads,

or, (*c*) it may be obtained by adding up the squares of the individual tetrads and taking their average.

* Using $\beta_4 = 5$, as corresponding to $\beta_1 = 0$, $\beta_2 = 2$ approximately.

Now (*c*) should give the exact value to any required number of places. (*a*) and (*b*) will differ from each other and (*c*) owing to the subrange elements adopted for grouping, and even if corrected by Sheppard may fail to give adequate accordance, i.e. their divergence may be in the same decimal place as the first significant figure of the probable error of $\sigma_T^2$. We will now investigate this difficulty.

(*a*) We formed the 378 tetrads and grouped them in subranges of ·01; the following distribution was obtained:

*Magnitude of Tetrads, taken as positive.*

| | ·00—·01 | ·01—·02 | ·02—·03 | ·03—·04 | ·04—·05 | ·05—·06 | ·06—·07 | ·07—·08 | ·08—·09 | ·09—·10 | ·10—·11 | ·11—·12 | ·12—·13 | ·13—·14 | ·14—·15 | ·15—·16 | ·16—·17 | ·17—·18 | ·18—·19 | ·19—·20 | ·20—·21 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 37 | 38 | 33 | 20 | 37 | 41·5 | 29·5 | 30 | 16 | 24 | 17 | 17 | 6 | 8 | 5 | 8 | 3 | 1 | 5 | 1 | 1 | 378 |

The constants of this distribution are as follows:

Mean (i.e. mean deviation from zero) ·062,341,

Median (Quartile of Symmetrical Distribution*) ·0570,

$\sigma_T$ = Standard Deviation (for Symmetrical Distribution) ·076,117,

$\mu_2 = \sigma_T^2 = ·0057,9392, \qquad \mu_3 = 0, \qquad \mu_4 = ·0000,8717,6119,$

and hence $\beta_2 = 2·5969.$

Is this a normal curve? Clearly not, if we are to judge by $\beta_2$, which should be 3. We should indeed expect a platykurtic and not a normal curve theoretically, for the tetrads must all lie numerically between $-1$ and $+1$, and thus must give a limited range symmetrical curve, and not a normal curve. Again the median ·0570 should be ·674,490 × ·076,117 = ·051,340 with a probable error ·001,851, or the observed median deviates more than three times its probable error from the value it should take on the basis of a normal curve (cf. xlviii). Again by (xlix) the mean should be ·060,733 ± ·002,182, the actual mean is ·062,341, and this test at least is satisfied.

Lastly we proceeded to apply the $(P, \chi^2)$ test for goodness of fit. Now we do not assert that this test is wholly suitable to material such as the present for it is based on random selection from an infinite population, any member of which is equally likely to be drawn. But when we are confronted with a graph and asked to approve the excellency with which the observations fit a normal curve of definite theoretical standard deviation, we naturally ask, failing to judge kurtosis by the eye: Should we consider in the ordinary treatment of observations such a series well fitted by a normal curve?

* The 189th tetrad of the system arranged in order is ·0567 and the 190th tetrad ·0574, giving ·05705 for the mid-value.

SYMMETRICAL DISTRIBUTION OF THE TETRAD DIFFERENCES. *HOLZINGER'S DATA.*

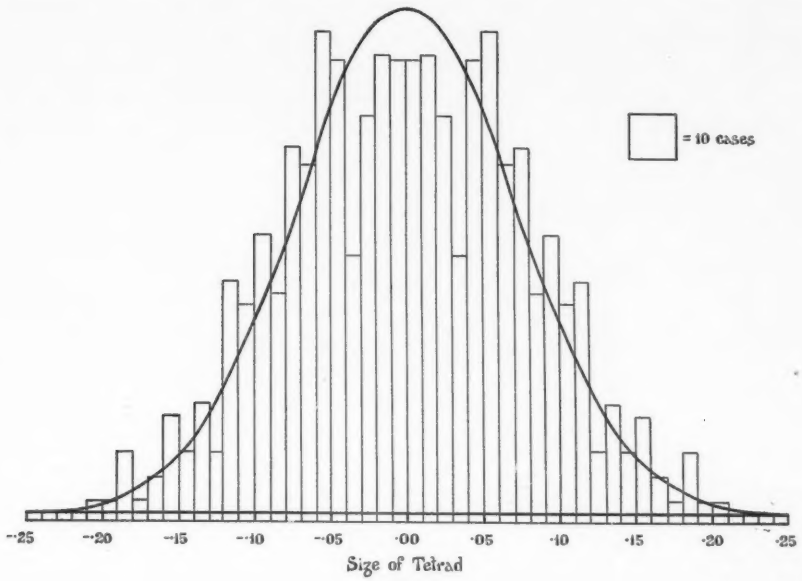


Diagram I.

Now the theoretical value of $\sigma_T$ is ·071,519 (see p. 274), and we have the following system of values:

*Fit of Holzinger's Data Tetrads to a Normal Curve.*

Ranges.

| Tetrad Values | ·00—·01 | ·01—·02 | ·02—·03 | ·03—·04 | ·04—·05 | ·05—·06 | ·06—·07 | ·07—·08 | ·08—·09 | ·09—·10 | ·10—·11 | ·11—·12 | ·12—·13 | ·13—·14 | ·14—·15 | ·15—·16 | ·16—·18 | ·18—·21 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Normal Curve $m$ | 42·0 | 41·2 | 39·7 | 37·4 | 34·6 | 31·4 | 27·9 | 24·3 | 20·8 | 17·5 | 14·4 | 11·6 | 9·2 | 7·1 | 5·4 | 4·0 | 5·1 | 4·5 | 378·1 |
| Observed Value $m'$ | 37 | 38 | 33 | 20 | 37 | 41·5 | 29·5 | 30 | 16 | 24 | 17 | 17 | 6 | 8 | 5 | 8 | 4 | 7 | 378 |
| $\frac{(m'-m)^2}{m}$ | ·60 | ·25 | 1·13 | 8·09 | ·17 | 3·25 | ·09 | 1·34 | 1·11 | 2·41 | ·47 | 2·51 | 1·11 | ·11 | ·03 | 1·00 | ·24 | 1·39 | $=\frac{\chi^2}{25·30}$ |

Here $\chi^2 = 25·30$ for 18 groups, giving a probability of $P = ·0886$. It must be noted that the chief contributions to $\chi^2$ do not arise from large values of the tetrads. Now the value of $P$ does not represent a great improbability, one sample

in 11 would give a fit as bad or worse to this normal curve, but we do think that the phrase "evidently the agreement could hardly be bettered" is unjustified either by the goodness of fit of the normal curve, or the correspondence between the observed median ·057* and the theoretical median, i.e. ·67449 × theoretical $\sigma_T$ (·071,519†) = ·048, the difference being about 4·7 times the probable error.

Even if these two values, one obtained from observation and the other from an approximate theory, were within the limits of random sampling as indicated by the probable error—and they are not in this case—it would only mean that *one* constant of a very intricate frequency distribution did not contradict an imperfect theory. Some tetrads might be nearly zero, others large, and the averaging process might easily bring the mean square tetrad within the limits of random sampling. Such agreement in a single constant of the distribution would never enable us to assert as Holzinger and Spearman do of the former's data: "that in this case at any rate *every one* of the abilities can be resolved into two independent factors, the one being always specific and the other throughout common‡."

Several illustrations of the danger of this type of argument occur in *The Abilities of Man*. Thus on p. 149, where Magson's data are dealt with, Professor Spearman writes: "Here the tetrad differences gave a probable error of ·03 and an observed median of ·04 so that again agreement leaves nothing to be desired."

As a matter of fact, taking these values as correct, ·01 may mean serious disagreement when, owing to the large number of individuals tested (149), the probable error of the median is found to be sensible only in the third or fourth decimal place.

Again, take the results of Bonser with 757 subjects and 20 correlations (*The Abilities of Man*, p. 147). Here Professor Spearman says that "the tetrad differences have a probable error of ·011 and an observed median of ·013, or almost exactly the same." Whether these numbers are "almost exactly the same" cannot be determined by inspection; it all depends on the probable error of their difference, and whether that error has its first significant figure in the third or fourth place of decimals. We will illustrate the validity of this criticism by giving the constants for Bonser's data. The standard deviation of the tetrads, $\sigma_T$ = ·019,942, the median as computed from this is ·013,451, while the quartile or observed median is ·019,056. The correlations give

$$\bar{r} = ·3423, \quad \sigma_r = ·083,826, \quad \mu_2 = ·0070,2681, \quad \mu_3 = -·0000,8836, \quad \mu_4 = ·0001,0313;$$

whence $\beta_1 = ·0023$, and $\beta_2 = 2·0887$, and a reasonable value of $\beta_4$ would be 5, say.

Hence we find $\quad \sigma_{\bar{r}}^2 = ·0003,2198, \quad \sigma^2_{\mu_2} = ·0000,5945$

and $\quad \{\delta\bar{r}\,\delta\mu_2\} = -·0000,0193;$

leading to $\quad \sigma_T = ·017,523, \quad \sigma_{\sigma_T^2} = ·0000,4861$

and $\quad \sigma_{\sigma_T} = ·001,387.$

* Spearman, *The Abilities of Man*, p. 149, gives the theoretical median which he terms "the probable error" as ·051 and the observed median as ·052. We are unable to agree with these values.

† The true value; the value ·076,117 used on the previous page to test normality is the $\sigma_T$ of the symmetrical distribution of tetrads.

‡ *British Journal of Psychology*, Vol. XVI. p. 88.

If we express our results in the somewhat unusual terminology of Professor Spearman, the observed median is ·013,451 and the theoretical probable error ·011,819 ± ·000,631. He gives these as ·013 and ·011; saying that they are "almost exactly the same." The actual difference is ·001,632, almost exactly 2·6 times the probable error of the difference.

We do not delay to deal with Magson's data, where the probable error will be significant only in the third or fourth place of decimals, while Spearman places before the reader values only to two places of decimals, remarking that their agreement leaves nothing to be desired.

(*b*) We next formed the distribution of square tetrads. This was done in subranges of ·001, and is reproduced in the following table:

*Distribution of Square Tetrads for Holzinger's Data.*

| Value of Squared Tetrad | Frequency | Value of Squared Tetrad | Frequency | Value of Squared Tetrad | Frequency |
|---|---|---|---|---|---|
| ·000—·001 | 109 | ·014—·015 | 6 | ·028—·029 | 2 |
| ·001—·002 | 40 | ·015—·016 | 5 | ·029—·030 | 1 |
| ·002—·003 | 35 | ·016—·017 | 2 | ·030—·031 | 1 |
| ·003—·004 | 31 | ·017—·018 | 5 | ·031—·032 | — |
| ·004—·005 | 22 | ·018—·019 | 3 | ·032—·033 | — |
| ·005—·006 | 22 | ·019—·020 | 1 | ·033—·034 | — |
| ·006—·007 | 9 | ·020—·021 | 2 | ·034—·035 | 3 |
| ·007—·008 | 12 | ·021—·022 | 1 | ·035—·036 | 2 |
| ·008—·009 | 15 | ·022—·023 | 1 | ·036—·037 | — |
| ·009—·010 | 10 | ·023—·024 | 2 | ·037—·038 | — |
| ·010—·011 | 9 | ·024—·025 | 3 | ·038—·039 | 1 |
| ·011—·012 | 8 | ·025—·026 | 4 | ·039—·040 | — |
| ·012—·013 | 4 | ·026—·027 | — | ·040—·041 | 1 |
| ·013—·014 | 6 | ·027—·028 | — | ·041—·042 | — |

The constants of this distribution are as follows:

$$\text{Mean} = \sigma_T^2 = \cdot 005{,}968, \qquad \Sigma = \text{Standard Deviation} = \cdot 007{,}465,$$

$$\mu_2 = \cdot 0000{,}5571{,}89, \qquad \mu_3 = \cdot 000{,}000{,}876{,}172, \qquad \mu_4 = \cdot 000{,}000{,}023{,}904{,}601;$$

whence $$\beta_1 = 4{\cdot}4376, \text{ and } \beta_2 = 7{\cdot}6995.$$

If the distribution of the simple tetrad were a normal curve, the values of $\beta_1$ and $\beta_2$ should be by (xlvii) 8 and 15 respectively. The observed values show how far this is from being the fact. Further we ought to have by (xlvi)

$$\Sigma = \sqrt{2}\,\sigma_T^2 = \sqrt{2} \times \cdot 005{,}968 = \cdot 008{,}440,$$

but the observed value is ·007,465.

We think these divergences, taken in conjunction with the previous results, are sufficient to indicate that the tetrads of the Holzinger data do not follow a normal curve. This does not disprove the Spearman hypothesis, because theoretical reasoning is all against such a distribution. But it does show that merely placing a normal curve on top of the tetrad histogram contributes nothing to the proof of the hypothesis.

(*c*) Lastly, we added the 378 squared tetrads together and took their mean, thus obtaining what should be the exact value of $\sigma_T^2$. We found

$$\sigma_T^2 = \cdot0059{,}4094.$$

Unfortunately in squaring our tetrads, which were themselves only available to four decimal places, the correlations only being given to two, we tabled the squares to *six* decimal places only and not to eight. It is possible therefore that this result is not correct to more than six decimal places.

We have now obtained three distinct values for $\sigma_T^2$ and of course for $\sigma_T$, and we see that they are by no means identical. Let us examine them.

*Values $\sigma_T^2$ and $\sigma_T$ by various processes.*

| Method | $\sigma_T^2$ | $\sigma_T$ |
|---|---|---|
| (*a*) From distribution of tetrads with subranges ·01 ... ... | ·0057,9392 | ·076,118 |
| (*b*) From distribution of Squared tetrads with subranges ·001 | ·0059,6825 | ·077,254 |
| (*c*) From addition of Squared tetrads to six decimal places ... | ·0059,4094 | ·077,077 |
| (*d*) From Formula (xvi) with Probable Error from (xxvii) ... | ·0051,1500 ± ·0003,9117 | ·071,519 ± ·002,736 |
| (*e*) From Formula (xxv) " " " " ... | ·0053,2495 ± ·0003,9117 | ·072,972 ± ·002,736 |

Now it will be clear from this table that the process by which we find the value of $\sigma_T^2$ is of vital importance, having regard to its probable error. The difference between the theoretical value of $\sigma_T^2$, as found by the full Formula (xvi) and as found by the Spearman-Holzinger approximation (xxv), is of the order of the probable error itself and this accordingly may make a considerable difference in the interpretation of our results. Again, the observed value of the mean squared tetrad differs according to whether we obtain it from the distribution of tetrads in ·01 subranges, or of squared tetrads in ·001 subranges, or by the simple adding together of the actual squared tetrads and dividing by their number. The third should give the most accurate value, the first being too coarse a grouping even when the standard deviation is corrected by Sheppard, and the second, although nearly in agreement with the third, probably owing to the *J*-shaped form of the distribution, requiring correction for abruptness. Taking (*d*) as the best theoretical value, and (*c*) as the best observed value, we have their difference = 2·11 times its probable error, while had we compared (*b*) and (*d*) the deviation would have been 2·18 times the probable error. If we consider the probable error in its ordinary sense, this is neither a very good nor a very bad accordance between theory and observation. What it certainly does not justify is the remark of Spearman and Holzinger that "in this case at any rate every one of the abilities can be resolved into two independent factors, the one being always specific and the other throughout common*."

* *Loc. cit.* p. 88. We have not considered it worth while to deal with the asymmetrical distribution of tetrads given in the paper, as it has been dropped by Spearman in *The Abilities of Man*, and clearly depends entirely on the order in which the variates are arranged in the table.

The results are about as doubtful if we consider $\sigma_T$ instead of $\sigma_T^2$, for the difference between (*c*) and (*d*) is then 2·03 times its probable error. If we multiply all the quantities in the $\sigma_T$ column by ·67449, we shall of course still have the difference as given by (*c*) and (*d*) 2·03 times its probable error. We do not get rid of this dubious value by using the less exact forms (*a*) and (*e*), and cutting them down to three figures and asking the reader to observe how close are ·052 and ·049!

Reduced to its simplest terms all we can say is that the difference between the mean squared tetrad as observed and as deduced on the hypothesis that the tetrads in the sampled population are really all zero is a little over twice its probable error. This is no convincing proof of the two-factor hypothesis, but it certainly does not disprove it, and this the more so as we really do not know what is the nature of the curve of distribution of $\sigma_T^2$ and whether the "probable error" has any meaning for that distribution at all. Nothing whatever is gained in perspicuity by reducing $\sigma_T^2$ first to $\sigma_T$ and then multiplying it by ·67449, unless we assume the distribution of tetrads to be normal and wish to compare the theoretical median tetrad from (*d*), i.e. ·048,239 ± ·001,845, with the observed median tetrad ·05705, when the difference will be found to be about 4·8 times the probable error.

That Formula (xxv) gives an unsatisfactory result as compared with the completer Formula (xvi) can be elucidated by comparing the terms of each order of magnitude in the two results. We have the following values:

*Holzinger's Data.*

| Order of Terms* | Holzinger-Spearman Approximation Formula (xxv) | Value from General Expression (xvi) |
|---|---|---|
| Square ... | ·0108,3956 | ·0108,3956 |
| Cubic ... | −·0066,6080 | −·0069,0538 |
| Quartic ... | ·0011,4616 | ·0011,8082 |
| Totals ... | ·0053,2492 | ·0051,1500 |

It will be seen that the Holzinger-Spearman approximation fails in the second significant figure in the cubic, and in the third significant figure in the quartic terms. That formula therefore tends to emphasise an accordance which does not really exist. This will be still more the case if the erroneous term $2(1-\bar{r}^2)^4/n^2$ be added in. The origin of this divergence of (xxv) from (xvi) lies in the erroneous estimates of the mean products involved: see our pp. 254—257.

* Considering the sources of (xxvi) we may break it up into

$$\frac{4}{n}(\bar{\rho}^2+\sigma_\rho^2), \quad -\frac{8\bar{\rho}}{n}\left(\bar{\rho}^2-\frac{3}{m-1}\sigma_\rho^2\right) \text{ and } \frac{4}{n}\left(\bar{\rho}^4-\frac{6\bar{\rho}^2\sigma_\rho^2}{m-1}\right),$$

representing the square, cubic and fourth order terms, where, of course, we must replace $\bar{\rho}$ and $\sigma_\rho$ by $\bar{r}$ and $\sigma_r$, the observed quantities. For the Holzinger data, $\bar{r}=$·347,778, $\sigma_r^2=$·0145,4500, $n=50$ and $m=36$.

While the Holzinger data give no such triumphant confirmation of the theory as they have been supposed to do,—for they only give a somewhat dubious answer to one needful, but far from all-sufficient condition, namely, the question whether the mean squared tetrad is compatible with selection from a population in which all the tetrads are zero,—the arithmetical analysis has brought out several important points. It shows us in the first place that at present an adequate expression to replace (xvi) by a briefer labour of computation does not exist; (xxv) introduces errors of the order of the probable error. In the second place, with the view of getting an adequately small error of random sampling even for the single condition involved, it is essential that $n$ should be far larger than the value provided by the Holzinger, or indeed by the Simpson data. And the larger $n$ is, the less attention need be paid to the unknown terms in $1/(n\sqrt{n})$ and $1/n^2$. Further, the Holzinger data show that the assumption that the tetrad distribution for the two-factor hypothesis must follow a normal curve is of no validity from the observational as well as from the theoretical standpoint, and thus to make it only obscures and does not strengthen the proof. We simply do not know what is the character of the distribution, even if the mental abilities concerned followed a normal law. Lastly, and this is a most vital point, it must be now clear that working to two or three decimal places, or at least stating final results to this number of places, is wholly inadequate, having regard to the decimal places in which the first significant figures of the probable errors appear. It may even be doubted whether correlations tabled to two decimal places only are really adequate, for the different processes which may be used to obtain the observed mean squared tetrad, $\sigma_T^2$, show to what a high degree of accuracy we must work, should we desire to obtain this constant correct to the fourth decimal place where its probable error begins to be appreciable*. This is again an argument in favour of a high value for $n$. We do not believe that anything like enough stress has thus far been laid on the need for an adequate number of decimal places in the computation of these data.

(12) *Illustration II. Simpson's Data.*

We now turn to Simpson's data, which involve 14 tests leading to 3003 tetrad differences. Unfortunately these tests were only applied to 37 persons, so that the probable errors of all the correlations are exceedingly large and as some 45 out of the 91 correlations are greater than ·50 and 24 greater than ·60, it is not easy to consider the correlations as following normal distributions, far less to assert that the individual tetrad differences would follow such distributions. However, the table of correlations is provided on p. 283.

We have treated these data precisely as we have treated those of Holzinger. We proceeded in the same manner to evaluate first the general Formula (lv), the work

* Suppose two correlations ·334 and ·754 involved in a tetrad are cut down to ·33 and ·75, their product in the latter case to six figures is ·172,500, whereas their real product is ·176,436. Thus the final tetrad may be modified in the third decimal place, and the squared tetrad quite possibly in this place also.

being in this case extremely long and laborious as there were 13 products for each of the 1001 triplets of tetrads. This led to

$$\Sigma S(\rho^2) = 1684{\cdot}0890,$$

$$\Sigma(P_3) = 551{\cdot}619{,}216, \quad \Sigma(P_4) = 204{\cdot}743{,}107.$$

Hence
$$\{\psi_r{}^2\} = \sigma_T{}^2 = \frac{877{\cdot}435{,}132}{3003 \times 37} = {\cdot}0078{,}9692,$$

and
$$\sigma_T = {\cdot}0888{,}6463.$$

*Simpson's Correlations**.

| | *a* | *b* | *c* | *d* | *e* | *f* | *g* | *h* | *i* | *j* | *k* | *l* | *m* | *n* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *a* | — | ·98 | ·94 | ·79 | ·62 | ·91 | ·71 | ·54 | ·78 | ·88 | ·55 | ·42 | ·33 | ·25 |
| *b* | ·98 | — | ·84 | ·80 | ·64 | ·81 | ·79 | ·70 | ·73 | ·74 | ·52 | ·43 | ·26 | ·25 |
| *c* | ·94 | ·84 | — | ·62 | ·55 | ·82 | ·49 | ·56 | ·73 | ·71 | ·53 | ·40 | ·28 | ·21 |
| *d* | ·79 | ·80 | ·62 | — | ·57 | ·52 | ·68 | ·53 | ·42 | ·56 | ·45 | ·29 | ·38 | ·48 |
| *e* | ·62 | ·64 | ·55 | ·57 | — | ·55 | ·54 | ·73 | ·39 | ·51 | ·39 | ·59 | ·25 | ·22 |
| *f* | ·91 | ·81 | ·82 | ·52 | ·55 | — | ·53 | ·57 | ·59 | ·66 | ·54 | ·31 | ·28 | ·19 |
| *g* | ·71 | ·79 | ·49 | ·68 | ·54 | ·53 | — | ·45 | ·39 | ·47 | ·51 | ·57 | ·17 | ·25 |
| *h* | ·54 | ·70 | ·56 | ·53 | ·73 | ·57 | ·45 | — | ·35 | ·49 | ·34 | ·56 | ·25 | ·25 |
| *i* | ·78 | ·73 | ·73 | ·42 | ·39 | ·59 | ·39 | ·35 | — | ·69 | ·36 | ·29 | ·26 | ·09 |
| *j* | ·88 | ·74 | ·71 | ·56 | ·51 | ·66 | ·47 | ·49 | ·69 | — | ·44 | ·37 | ·34 | ·28 |
| *k* | ·55 | ·52 | ·53 | ·45 | ·39 | ·54 | ·51 | ·34 | ·36 | ·44 | — | ·31 | ·19 | ·27 |
| *l* | ·42 | ·43 | ·40 | ·29 | ·59 | ·31 | ·57 | ·56 | ·29 | ·37 | ·31 | — | ·21 | ·07 |
| *m* | ·33 | ·26 | ·28 | ·38 | ·25 | ·28 | ·17 | ·25 | ·26 | ·34 | ·19 | ·21 | — | ·24 |
| *n* | ·25 | ·25 | ·21 | ·48 | ·22 | ·19 | ·25 | ·25 | ·09 | ·28 | ·27 | ·07 | ·24 | — |

To find the probable error of $\sigma_T$ we need first to consider the distribution of the correlation coefficients, 91 in number. The constants obtained by direct summation without grouping are as follows:

$$\bar{r} = {\cdot}486{,}703, \quad \sigma_r = {\cdot}208{,}617,$$

$$\mu_2 = {\cdot}0435{,}2100, \quad \mu_3 = {\cdot}0024{,}2837, \quad \mu_4 = {\cdot}0043{,}7599,$$

$$\beta_1 = {\cdot}0715, \quad \beta_2 = 2{\cdot}3104,$$

a slightly skew, but very platykurtic distribution.

Accordingly from (xxv):

$$\sigma_T{}^2 = \tfrac{4}{37}[{\cdot}0624{,}1163 + {\cdot}0442{,}4584 - {\cdot}0000{,}0004 + {\cdot}0000{,}0070],$$

where the magnitude of the several terms is indicated. Hence

$$\sigma_T{}^2 = {\cdot}0115{,}3047.$$

If we had omitted the last two terms as of small importance, the value would have been ·0115,3054†. The former yields $\sigma_T = {\cdot}1073{,}8003$.

* *Psychological Review*, 1914, pp. 101 *et seq.* The *hk* correlation is in one instance given as ·54 and in the other as ·34. We have followed Spearman, who gives them both as ·34 in *The Abilities of Man*, p. 145.

† This is the value given by the Spearman-Holzinger formula when we discard their erroneous term in $1/n^2$.

We are now in a position to compute the auxiliary quantities on which the standard deviations of $\sigma_T^2$ and $\sigma_T$ depend. We find, remembering $p=14$, $m=91$, and $n=37$, that

$$\sigma_{\bar{r}}^2 = \cdot 0041{,}96325, \quad \sigma^2_{\mu_2} = \cdot 0000{,}37195^*,$$

$$\{\delta\bar{r}\,\delta\mu_2\} = -\cdot 0000{,}34181.$$

Further:
$$\bar{r}(1-\bar{r})(1-2\bar{r}) + \frac{3(1-2\bar{r})}{m-1}\mu_2 = \cdot 0066{,}8238,$$

$$1 + \frac{6\bar{r}(1-\bar{r})}{m-1} = 1\cdot 0166{,}5488.$$

Whence
$$\sigma_{\sigma_T^2} = \cdot 0006{,}68744,$$

and
$$\sigma_{\sigma_T} = \sigma_{\sigma_T^2}/(2\sigma_T) = \cdot 0037{,}62712,$$

the corresponding probable errors being

$$\text{P.E. of } \sigma_{\sigma_T^2} = \cdot 0004{,}5106, \quad \text{P.E. of } \sigma_{\sigma_T} = \cdot 0025{,}3791.$$

We have now to compare the three quantities $\sigma_T^2$ as found from (lv), $\sigma_T^2$ as found from (xxv), and the observed values as found from the distributions of simple and squared tetrads. As in the case of Holzinger's data we provide first a table showing the contributions of (lv) and (xxv) to the terms of square, cubic, and quartic order in $\sigma_T^2$.

*Simpson's Data.*

| Order of Terms | Holzinger-Spearman Approximation Formula (xxv) | Value from General Expression (xvi) or (lv) |
|---|---|---|
| Square ... | ·0303,1361 | ·0303,1363† |
| Cubic ... | − ·0247,7493 | − ·0297,8747 |
| Quartic ... | ·0059,9187 | ·0073,7076 |
| Totals ... | ·0115,3055‡ | ·0078,9692 |

These results, while sufficiently startling, really only confirm the conclusion reached on the Holzinger data, namely, that the Holzinger-Spearman approximate formula fails to provide the cubic and quartic terms with anything like the required degree of accuracy. It may fail in the second or even in the first significant figure. This is the more important as in this case the cubic terms are practically as large as the square terms. It would accordingly seem absolutely necessary to use the full Formula (xvi) or (lv), although our painful experience of

* Adopting the value $\beta_4 = 7\cdot 8$: see footnote, p. 260.

† This should agree absolutely with figure in first column; the difference of two in the eighth decimal place is due to number of figures retained in different stages of the working.

‡ Agrees with the exception of one unit in last figure with value found on p. 283.

the tremendous computing labour involved makes us indeed reluctant to accept this as a final conclusion. What appears certain is that the Holzinger-Spearman formula will not give the required result, and the explanation of this no doubt lies in their assumption that $[X_1X_2X_3\ldots]$ is the same as $\{X_1X_2X_3\ldots\}$: see our pp. 254—7.

We have then for our theoretical values:

$$\sigma_T^2 = \cdot0078{,}9692 \pm \cdot0004{,}5106,$$
$$\sigma_T = \cdot0888{,}6463 \pm \cdot0025{,}3791.$$

It is clear that the Holzinger-Spearman formula gives a result for $\sigma_T^2$ differing more than eight times the probable error from the exacter value of Formula (lv)!

We now pass to the consideration of the frequency distributions of the tetrads.

For the distribution of tetrads treated as positive we find:

*Simpson's Data.*

| Range of Tetrad | ·00—·02 | ·02—·04 | ·04—·06 | ·06—·08 | ·08—·10 | ·10—·12 | ·12—·14 | ·14—·16 | ·16—·18 | ·18—·20 | ·20—·22 | ·22—·24 | ·24—·26 | ·26—·28 | ·28—·30 | ·30—·32 | ·32—·34 | ·34—·36 | ·36—·38 | ·38—·40 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency | 580·5 | 590·5 | 456·5 | 354·5 | 310 | 162·5 | 145·5 | 98 | 58 | 44 | 49 | 40 | 37 | 19 | 26 | 17 | 5 | 6 | 3 | 1 | 3003 |

The constants of this distribution are:

Mean Tetrad = ·073,493, Standard Deviation, $\sigma_T = \cdot099{,}090$,

$\mu_2 = \cdot0098{,}1874$*, $\mu_4 = \cdot0004{,}2416{,}8640$,

leading to $\beta_2 = 4{\cdot}2778$.

It is thus already clear that the distribution is far from normal ($\beta_2 = 3$). Also we have in this case to deal with a leptokurtic and not a platykurtic curve. This result confirms a conclusion forced on us by the examination of all the material used to discuss the two-factor hypothesis, namely, that the actual nature of the tetrad distribution curve depends on the special mental characters chosen to provide the material, and there is no universal rule that the tetrad distribution will form a curve of a particular character. We conceive it quite possible that with a special selection of mental abilities a tetrad distribution might be bimodal.

With a $\beta_2$ as divergent as these data indicate, there is small probability of the tetrad distribution showing any goodness of fit to a normal curve. Taking the theoretical standard deviation as ·08886, we have worked out the frequencies in 18 groups of the corresponding normal curve. This curve is shown superposed on

* Without Sheppard's correction $\mu_2 = \cdot0098{,}2707$ and $\sigma_T = \cdot099{,}132$; the differences are scarcely of importance with 20 groups.

the observations in Diagram II, and the following table provides the corresponding frequencies.

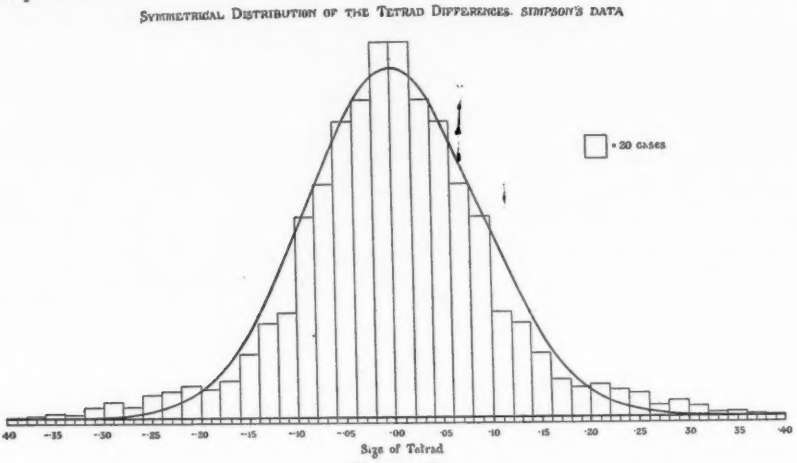
Symmetrical Distribution of the Tetrad Differences. Simpson's Data
= 20 cases
-40 -35 -30 -25 -20 -15 -10 -05 ·00 ·05 ·10 ·15 ·20 ·25 ·30 ·35 ·40
Size of Tetrad

Diagram II.

Now we are quite certain that it is not possible to appreciate goodness of fit by examining ocularly the above diagram, but we can say that if the tetrad distribution were a series of ordinary observations obtained by random selection from a normal curve, not once in 10,000,000 trials should we obtain as bad or worse a fit.

*Fit of Simpson's Data Tetrads to a Normal Curve.*

Ranges.

| Tetrad Values | ·00—·02 | ·02—·04 | ·04—·06 | ·06—·08 | ·08—·10 | ·10—·12 | ·12—·14 | ·14—·16 | ·16—·18 | ·18—·20 | ·20—·22 | ·22—·24 | ·24—·26 | ·26—·28 | ·28—·30 | ·30—·32 | ·32—·34 | ·34—·40 | T |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Normal Curve $m$ | 534·8 | 508·1 | 459·9 | 395·2 | 322·9 | 251·5 | 185·4 | 130·3 | 87·0 | 55·3 | 33·3 | 19·1 | 10·5 | 5·4 | 2·7 | 1·3 | 0·5 | 0·4 | 3 |
| Observed Value $m'$ | 580·5 | 590·5 | 456·5 | 354·5 | 310 | 162·5 | 145·5 | 98 | 58 | 44 | 49 | 40 | 37 | 19 | 26 | 17 | 5 | 10 | 3 |
| $\frac{(m'-m)^2}{m}$ | 3·91 | 13·36 | ·03 | 4·19 | ·51 | 31·50 | 8·59 | 8·01 | ·97 | 2·31 | 7·40 | 22·87 | 52·12 | 34·25 | 201·07 | 189·61 | 40·50 | 230·40 | 8 |

And this result does not flow from the large tetrads at the tail only. If we take only tetrads from ·00 to ·16, or eight groups with $\chi^2 = 90{\cdot}10$, then not once in a million trials should we have such a bad fit! Now these probabilities might to some extent be modified could we take account of the intercorrelation of the variants. But it is clear that no existing means, and with the highest degree of probability no revised process, could lead to anything more than the statement that it is very unreasonable to suppose these tetrads follow in their distribution a normal curve of errors.

If further reasoning be necessary on this point we can refer to the following constants.

The actually observed median tetrad is ·0540; it should be ·059,938 on the theoretical standard deviation and ·066,835 on the observed standard deviation. The deviation of the observed median is about 3·5 times its probable error in the first case and about 7·5 times its probable error in the second case.

The observed mean tetrad is ·073,493; deduced from the theoretical standard deviation it is ·070,904 with a probable error ± ·002,024; deduced from the observed standard deviation it is ·079,062. The observed mean tetrad differs from its theoretical value only 1·2 times the probable error, indicating that the actual curve might be a normal curve with the theoretical standard deviation; but the observed mean tetrad differs from the mean deduced from the observed standard deviation by more than four times this probable error, indicating that the actual curve is not a normal curve at all. No result from the distribution of positive tetrads seems compatible with any real approach to normality.

We next take the Distribution of Squared Tetrads given in the following table:

*Distribution of Square Tetrad Differences in the Correlations of Simpson's Observations.*

| Value | Freq. | Value | Freq. | Value | Freq. | Value | Freq. | Value | Freq. |
|---|---|---|---|---|---|---|---|---|---|
| ·000—·001 | 932 | ·031—·032 | 1 | ·062—·063 | 6 | ·093—·094 | 2 | ·124—·125 | 1 |
| ·001—·002 | 351 | ·032—·033 | 8 | ·063—·064 | 2 | ·094—·095 | 2 | ·125—·126 | — |
| ·002—·003 | 235 | ·033—·034 | 8 | ·064—·065 | 4 | ·095—·096 | 2 | ·126—·127 | — |
| ·003—·004 | 176 | ·034—·035 | 8 | ·065—·066 | 3 | ·096—·097 | 2 | ·127—·128 | 1 |
| ·004—·005 | 144 | ·035—·036 | 3 | ·066—·067 | 3 | ·097—·098 | 2 | ·128—·129 | — |
| ·005—·006 | 99 | ·036—·037 | 8 | ·067—·068 | 4 | ·098—·099 | 2 | ·129—·130 | — |
| ·006—·007 | 104 | ·037—·038 | — | ·068—·069 | 2 | ·099—·100 | 1 | ·130—·131 | — |
| ·007—·008 | 106 | ·038—·039 | 5 | ·069—·070 | 3 | ·100—·101 | 1 | ·131—·132 | — |
| ·008—·009 | 83 | ·039—·040 | 9 | ·070—·071 | 1 | ·101—·102 | — | ·132—·133 | — |
| ·009—·010 | 61 | ·040—·041 | 7 | ·071—·072 | 1 | ·102—·103 | — | ·133—·134 | — |
| ·010—·011 | 46 | ·041—·042 | 9 | ·072—·073 | 3 | ·103—·104 | — | ·134—·135 | 2 |
| ·011—·012 | 46 | ·042—·043 | 5 | ·073—·074 | 4 | ·104—·105 | 1 | ·135—·136 | — |
| ·012—·013 | 40 | ·043—·044 | 6 | ·074—·075 | 1 | ·105—·106 | 1 | ·136—·137 | 1 |
| ·013—·014 | 19 | ·044—·045 | 4 | ·075—·076 | 2 | ·106—·107 | — | ·137—·138 | — |
| ·014—·015 | 34 | ·045—·046 | 3 | ·076—·077 | — | ·107—·108 | — | ·138—·139 | — |
| ·015—·016 | 29 | ·046—·047 | 6 | ·077—·078 | 1 | ·108—·109 | 1 | ·139—·140 | — |
| ·016—·017 | 29 | ·047—·048 | 5 | ·078—·079 | 5 | ·109—·110 | — | ·140—·141 | — |
| ·017—·018 | 25 | ·048—·049 | 5 | ·079—·080 | 2 | ·110—·111 | — | ·141—·142 | — |
| ·018—·019 | 25 | ·049—·050 | 4 | ·080—·081 | — | ·111—·112 | — | ·142—·143 | — |
| ·019—·020 | 22 | ·050—·051 | 1 | ·081—·082 | 5 | ·112—·113 | — | ·143—·144 | — |
| ·020—·021 | 18 | ·051—·052 | 10 | ·082—·083 | 3 | ·113—·114 | — | ·144—·145 | — |
| ·021—·022 | 21 | ·052—·053 | 3 | ·083—·084 | 3 | ·114—·115 | 2 | ·145—·146 | — |
| ·022—·023 | 15 | ·053—·054 | 6 | ·084—·085 | 3 | ·115—·116 | 1 | ·146—·147 | — |
| ·023—·024 | 15 | ·054—·055 | 6 | ·085—·086 | 1 | ·116—·117 | — | ·147—·148 | — |
| ·024—·025 | 14 | ·055—·056 | 4 | ·086—·087 | 1 | ·117—·118 | 1 | ·148—·149 | — |
| ·025—·026 | 10 | ·056—·057 | 4 | ·087—·088 | 3 | ·118—·119 | 1 | ·149—·150 | — |
| ·026—·027 | 10 | ·057—·058 | 4 | ·088—·089 | — | ·119—·120 | — | ·150—·151 | — |
| ·027—·028 | 12 | ·058—·059 | 2 | ·089—·090 | 1 | ·120—·121 | — | ·151—·152 | 1 |
| ·028—·029 | 9 | ·059—·060 | 5 | ·090—·091 | — | ·121—·122 | — | | |
| ·029—·030 | 8 | ·060—·061 | 4 | ·091—·092 | 2 | ·122—·123 | — | | |
| ·030—·031 | 9 | ·061—·062 | 5 | ·092—·093 | 1 | ·123—·124 | — | Total ... | 3003 |

The constants of this table are as follows:

Mean Square Tetrad = ·009,866, Standard Deviation, $\sigma_T^2$ = ·018,002,

$\mu_2$ = ·000,324,066,286, $\mu_3$ = ·000,019,282,816, $\mu_4$ = ·000,001,651,732.

These give $\beta_1 = 10{\cdot}9255$, $\beta_2 = 15{\cdot}7279$.

Constants to be expected for the Distribution of Mean Square Tetrads had the Distribution of Positive Tetrads been a Normal Curve with the Theoretical Standard Deviation:

| | *Expected* | *Observed* |
|---|---|---|
| Mean: | ·007,897 | ·009,866 |
| Standard Deviation: | ·011,168 | ·018,002 |
| $\beta_1$: | 8 | 10·93 |
| $\beta_2$: | 15 | 15·73 |

We are again forced to the conclusion that the observed values are not such as we should expect to arise were the distribution of tetrads a normal curve of the theoretical standard deviation.

Thus the conclusion already drawn is again confirmed, the observations do not support the view that a normal curve of tetrad distribution is in any way an approximation in the case of Simpson's data.

Having dismissed the normal curve we are left with a *single* test of whether Simpson's data are compatible with the General and Specific Factors Hypothesis. That test is summed up in asking whether the difference between the theoretical and observed values of the Mean Squared Tetrad falls within the limits of random sampling. The Theoretical Mean Squared Tetrad is

$$·0078{,}9692 \pm ·0004{,}5106.$$

The Observed Mean Squared Tetrad as deduced from the standard deviation of grouped positive tetrads is ·0098,1874, and as deduced from the mean of grouped squared tetrads ·0098,6630. We have not so far had the courage to add up the 3003 squared tetrads and find their mean exactly, but we believe the mean obtained by grouping in ranges of ·001 must be fairly approximate, and the deviation from the observed value is then about 4·26 times the probable error, or the odds are about 250 to 1 against such a deviation. Notwithstanding that we know nothing of the distribution curve for Mean Squared Tetrads* it is fairly safe to assert that a deviation of this magnitude is wholly incompatible with the mean squared tetrad being zero in the sampled population—in other words it is quite incompatible with the two-factor hypothesis.

How then does it come about that Professor Spearman can write (*The Abilities of Man*, p. 146): "This time, the two distributions, curve and rectangles, far from

* If this material were of the ordinary kind, i.e. a sample of 3003 individuals made from a curve with constants similar to those of the distribution of squared tetrads, then the betas of the curve of means would be $B_1 = \frac{\beta_1}{3000}$, $B_2 - 3 = \frac{\beta_2 - 3}{3003}$, and notwithstanding the high values of $\beta_1$ and $\beta_2$, i.e. 10·93 and 15·73, we should have $B_1$ = ·0036 and $B_2$ = 3·0042, values compatible with a normal distribution of the means of mean squared tetrads, and so giving the usual significance to "probable error."

being totally discrepant as before, display instead one of the most striking agreements between theory and observation ever recorded in psychology. Indeed it would not easily be matched in any other science. The divisibility then is indicated more decisively than ever."

We venture to think that he has been misled by one or two unappreciated facts:

(*a*) The general symmetry is produced artificially by treating each tetrad as alternately positive and negative.

(*b*) That he lays weight on ocular judgment of goodness of fit where in every case the more certain analytical tests show no approach whatever to a normal distribution.

(*c*) That so far as theory is concerned it undoubtedly indicates that normality of distribution is highly improbable. (Professor Spearman gives no proof for it, nor does he state why he anticipates it.)

(*d*) A spurious appearance of the whole series of tetrads being close to zero is produced by an erroneous scale of one-tenth the true value of the tetrads being accidentally placed at the foot of the diagram.

(*e*) The results reached by him and printed underneath the diagram are:

$$\textit{Probable Error} = \cdot 061.$$
$$\textit{Observed Median} = \cdot 062.$$

Besides the unacceptable cutting down to three figures we believe these results to be both erroneous.

If by "Probable Error" is meant the theoretical standard deviation, $\sigma_T$, multiplied by ·674,490, then its value is ·0888,6463 × ·674,490 = ·059,938, or to three decimals ·060, so that the first result is somewhat in error. If by *Observed Median* Professor Spearman understands as in the usual terminology the value of the middle tetrad, this is ·0540, and differs widely from ·062. If by *Observed Median* Professor Spearman understands contrary to the usual terminology the value he would obtain from the observed $\sigma_T \times$ ·674,490, he is assuming that his distribution is truly normal, for which all proof fails, but its value is then

$$\cdot 099{,}090 \times \cdot 674{,}490 = \cdot 066{,}835,$$

or cut down to three figures ·067, whereas Professor Spearman gives ·062 as his *Observed Median*. Whether we take ·054 or ·067 as the *Observed Median* tetrad, the latter being obtained by unproven assumption, the difference from ·060 is some four times the probable error ·001,712 of the result. Thus the one point on which "one of the most striking agreements between theory and observation ever recorded in psychology" turns appears to be due to some oversight in arithmetic, and to the unappreciated fact that we cannot talk of agreement between two numbers in statistical science until we have some idea of the magnitude of the probable error of their difference.

Had indeed the difference in the *Observed Median* and the *Probable Error* been in the cases dealt with by Professor Spearman of the order of random sampling, it would not in the least have proved that the divisibility into two factors was decisively established. All it would have achieved would have been to show that in respect of one constant—an undoubtedly important but not all-sufficient constant, namely the mean square tetrad—the data were *compatible* with the hypothesis. Compatibility of data for *one* condition only with an hypothesis does not demonstrate the truth of the hypothesis; other hypotheses might show compatibility in this or other important conditions. As a matter of fact Holzinger's data only show dubious compatibility, and Simpson's data incompatibility; both, we hold, have been erroneously interpreted.

(13) *General Conclusions.*

We may sum up our conclusions as follows:

(*a*) The theory of general and specific factors is, we are inclined to believe, too narrow a structure to form a frame for the great variety of interrelationship of mental abilities. Still it is a theory which it is well worth while to test adequately.

(*b*) The mathematics of the theory—even if we confine ourselves to the narrow hypothesis of linearity adopted by Professor Spearman—become of surprising complexity, and the fundamental formula for the mean square tetrad involves for adequate data weeks of arithmetic labour to compute the resulting value.

(*c*) The formula provided by Holzinger and Spearman to avoid this labour is in error, and this for two reasons. In the first place, to get over the difficulty of testing only a small number of persons, they have endeavoured to evaluate the terms of the second order; their expression for these terms is in error because they are still using in the larger portion of their formula terms only computed to the first order of approximation. Secondly, their evaluation of the mean products of correlation coefficients fails because they have overlooked the fact that their values for mean products involve *all* correlation products, whereas actually only certain of these products occur in the theoretical mean square tetrad formula. We have illustrated how great may be the divergence from the true value found by computing the correlation products by their formula and by the full formula.

(*d*) The actual deduction of the probable error of the full mean squared tetrad formula presents great algebraic complexity, and if this could be overcome, it would almost certainly involve more serious arithmetic work than even that necessary to determine the mean square tetrad. The roughly approximate formula for the mean square tetrad of Holzinger and Spearman does, however, provide a basis upon which some estimate of the probable error of the mean square tetrad can be established. The process involves considerable algebraic and arithmetic labour, and one or two theoretical assumptions not yet justified. But in this way we reach the only method yet suggested for testing the compatibility of theory and observation. Applied, we find dubious compatibility in the case of Holzinger's data, even more dubious results in the case of Bonser's data, and what looks like

complete incompatibility for Simpson's data. Thus the claim of Professor Spearman to have effected a Copernican revolution in psychology seems at present premature.

(*e*) A satisfactory demonstration of his hypothesis would really require a knowledge of the theoretical values of the constants of the tetrad distribution curve, and the comparison of these by aid of their probable errors with the observed values. Even after the present investigation we must admit that we only know one constant and its approximate probable error.

Professor Spearman has surmounted the difficulty by assuming without any demonstration whatever that the distribution of tetrads rendered symmetrical follows a Gaussian or normal curve. There are grave theoretical reasons against this assumption. Since the correlation coefficient, especially for small samples, does not follow a normal curve, even when the variates follow a normal surface, there is no likelihood that a tetrad formed of four intercorrelated coefficients will do so. Still less likely is it that a number of correlated tetrads, each single one picked out from a distribution non-normal and with its own standard deviation (to which it is not even reduced!), will do so. The improbability of a tetrad distribution being a normal curve is illustrated on Holzinger's and again on Simpson's data. The former distribution is platykurtic and the latter leptokurtic, and this suggests, if it fails to prove, that each group of mental abilities will have its own peculiar type of distribution of tetrads.

(*f*) Dealing at length with Holzinger's data and Simpson's data we fail to find in them any adequate proof whatever that the agreement of the observed values with those required by theory is "surprisingly close," or that "In general it seems quite as good as, if not better than, that usually reached in determining the mechanical equivalent of heat and thus establishing the law of conservation of physical energy*."

Yet while very seriously doubting the validity of the arguments, both the accordances said to exist between theory and observation and the mathematical treatment of Professor Spearman, we believe that the hypothesis of a general factor should be further tested on new material. The nature of the non-overlapping mental abilities should be selected by psychological consensus before the tests are made, the number of persons tested should be far greater than in the case of Holzinger's or Simpson's data, trained computers should be employed, and if possible a more adequate mathematical theory of the whole subject developed. Then we might have a better chance either of dismissing the whole theory, or showing that it was worth while spending further energy in developing it. At present we can only return, but return definitely, a verdict of non-proven.

* *The Abilities of Man*, p. 160.

# GOITRE IN ADOLESCENCE; AN ANTHROPOMETRIC STUDY OF THE RELATION BETWEEN THE SIZE OF THE THYROID GLAND AND PHYSICAL AND MENTAL DEVELOPMENT.

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## (1) Foreword.

This research was undertaken in an endeavour to throw some light on the following problems:

(i) What is the best method of measuring the thyroid gland?

(ii) How does the size of the gland change about the period of adolescence?

(iii) What significance can be attached to the so-called "physiological enlargement" often noticed at this period?

(iv) Is there any real distinction between these slight enlargements and so-called "goitre" or is the difference merely one of degree?

(v) What relation exists between the size of the gland and physical and mental development about the adolescent period?

(vi) What association, if any, has the size of the gland with blood pressure and pulse rate?

I do not think that the voluminous literature on the thyroid gland supplies a really satisfactory answer to any of these questions. There is no doubt that too

little attention has been paid to the behaviour of the normal thyroid during growth and to the initial stages of goitre. There is a vague idea that a small degree of enlargement is to be regarded as a natural physiological reaction, in girls at least, during adolescence, and it is generally believed that most of these enlargements are evanescent. A medical man who has worked all his life in parts of Devonshire, Somersetshire, Lancashire or Cheshire will have naturally come to regard most adolescent thyroïd enlargements in girls as physiological because he sees them so commonly. And yet one has only to go into a London school and search for enlarged thyroids and then into a school at Stockport or Exeter to become immediately convinced that this physiological reaction, unless it can be accounted for by racial differences, must be a reaction to some external *local* factor and not to factors inherent in the organism itself.

If we admit that, it behoves us to determine whether the presence or absence of this reaction denotes some fault in metabolism, arising from local causes, which may have some influence on normal development, for although we may be satisfied that small enlargements are not associated with symptoms of ill-health we also need to satisfy ourselves that their presence or absence does not imply retardation of full physical or mental development. This is of more than academic interest because, leaving the vexed question of the causation of goitre on one side, we have at our disposal, if necessary, a method of largely reducing the incidence of thyroid enlargement in children, and evidence of the influence of this prophylactic administration of iodine on physical growth also has been provided in a recent paper (1).

I wish here to express my thanks to the Medical Department of the Board of Education, to the County Medical Officers of Health of Cheshire, Devonshire, Somersetshire, Denbighshire and Surrey, to the Medical Officers of Health of Stockport, Taunton, Bridgwater, Exeter, Barnstaple, Torquay, Plymouth, Bath and Tiverton and to the School Medical Department of the London County Council for kindly giving me facilities for this work, and also to the Head Teachers of the many schools visited for their ungrudging assistance in the carrying out of the actual measurements. The work of measuring and testing 440 girls in Cheshire schools was shared by my brother, Dr A. Vernon Stocks, whose name therefore appears at the head of this paper. I am also indebted to Professor Karl Pearson for many fruitful suggestions and to Miss McLearn for drawing the diagrams.

## (2) METHODS OF MEASUREMENT.

(*a*) *Introductory.* The first problem lay in the choice of a suitable and convenient method of measuring the size of the thyroid gland. The Swiss Goitre Commission lay down in their *Instruction concernant la Mensuration du corps thyroïde* (Bern, 1922) a division into five categories which may be roughly defined as follows: Category 0, gland not palpable; Category I, can be palpated but outlines are not visible in any position of the neck; Category II, outlines of gland readily seen with head thrown back or on swallowing but enlargement insufficient

to seriously alter the profile of the neck; Category III, sufficient enlargement to produce an evidently "full neck"; Category IV, undoubted goitre. Such a scheme, as they pointed out, has certain advantages over actual mensuration because (i) there is a fraction of cases in which the gland cannot be felt at all, or so imperfectly that its boundaries are difficult to define with accuracy, and (ii) when the gland is soft in consistence measurements depend to some extent on the pressure exerted by the measuring instrument.

The first objection to mensuration is not a serious one for our purpose because only a small proportion of children at ages over 10, in this country at any rate, have impalpable thyroids; in my own English series of 850 girls aged 11 and upwards actually measured I met with 20 such. At earlier ages it would be a more serious objection. In forming any distribution on a scale of measurement this small group can be suitably disposed of at the beginning of the scale. The second objection does not in my experience lead to serious error, particularly if the linear measurement of thyroid breadth be used; it may render difficult the measuring of height of the lobes, but as I shall presently show the use of this latter measurement is generally unnecessary.

The possible methods of actual measurement are:

(i) the circumference at the base of the neck, passing the tape over the spinous process of the 7th cervical vertebra behind and the hollow above the sternum in front;

(ii) the maximum antero-posterior diameter measuring from the spinous process of the 7th cervical vertebra;

(iii) the maximum breadth of the gland, measured by callipers;

(iv) the height of each lobe;

(v) Hunziker (2) has made use of the product of the last two measurements as a measure of the superficial area covered by the gland, taking the mean height of the two lobes as the vertical measurement when these are unequal.

I have used all these methods during this research in order to determine which is the most satisfactory for general use and for establishing standards. The results of the comparison of methods will now be discussed.

(*b*) *Estimation of thyroid size by visual judgment.* In a survey of school children aged 12 made by the School Medical Department of the Board of Education in 1924 the following definition of thyroid enlargement was used: "children in whom the thyroid is sufficiently enlarged for the increase in the size of the neck to be noticed on casual inspection without measurement or palpation." In order to correlate the present work with an analysis of the results of that survey, which will be published elsewhere (9), it was essential to make one of the boundaries between the categories I used in this work coincide with my own idea of the beginning of enlargement according to the above definition. I therefore used four categories as follows:

Category 01 { Category 0, comprising thyroids which could not be palpated, identical with Category 0 of the Swiss Goitre Commission.
Category 1, comprising thyroids which were palpable but presented no visible enlargement on inspection.

Category 2, comprising thyroids which showed visible enlargement, but not sufficient to produce gross deformity in the shape of the neck.

Category 3, comprising thyroids enlarged sufficiently to be termed "goitre" beyond question.

The boundary between 0 and 1 is definite enough, but the numbers found in Category 0 were too small to deal with separately, so the first two categories are usually referred to together as "Category 01." The boundary between 1 and 2 coincides with the limit set for the Board of Education's Medical Officers, and I have proved it to be subject to a large personal equation for different observers; my own interpretation of it coincides with my interpretation of the boundary between Categories I and II of the Swiss Goitre Commission. The boundary between Categories 2 and 3 is necessarily vague and would be subject to a still larger personal equation; the same applies however to the boundaries between II, III and IV of the Swiss Goitre Commission which I find difficult to interpret in practice. To summarize, we have the following scheme of comparison:

| Categories used by author | Definition used in Board of Education's Survey, 1924 | Swiss Goitre Commission's Categories |
|---|---|---|
| 01 { 0 | "Normal" | 0 |
| 01 { 1 | "Normal" | I |
| 2 | "Enlarged" | II |
| 2 | "Enlarged" | III |
| 3 | "Enlarged" | IV |

My own division into three groups corresponds to what I should at the outset of this work have regarded as "Normal" thyroids, "Physiological enlargements," and "Goitres," without attaching any real significance to the distinction between the last two.

A possible advantage of a method of visual judgment over actual measurements would be that the size of the child might be taken into consideration in forming a judgment; that is to say, it might be expected that the eye would judge of the apparent size of the gland not as an absolute measure but in comparison with the size of the neck, and that on the average it would be found that the limiting size of thyroid at which enlargement appeared to begin would increase

in proportion to the child's height. If this were the case the method would offer distinct advantages since correction for height would be unnecessary. Although my personal impression was that I was unconsciously correcting for the size of the child in this way when classifying by categories, analysis of the results showed that this impression was fallacious. The mean heights of the groups were as follows:

TABLE I.

| | | Mean Height | Standard Deviation | Probable Error |
|---|---|---|---|---|
| *Boys.* | Category 01 group ... | 139·27 cm. | 7·501 cm. | ± ·29 |
| | Categories 2—3 group ... | 141·26 „ | 7·318 „ | ± ·35 |
| | Difference ... ... | 1·99 „ | — | ± ·45 |
| *Girls.* | Category 01 group ... | 140·88 cm. | 7·842 cm. | ± ·62 |
| | „ 2 „ ... | 142·19 „ | 6·868 „ | ± ·30 |
| | „ 3 „ ... | 144·79 „ | 6·022 „ | ± ·49 |
| | Difference 01 and 3 ... | 3·91 „ | — | ± ·80 |

These differences are significant and the correlation coefficients* with height in boys and girls respectively are ·1658 ± ·0373 and ·1137 ± ·0302. In Section (5) it is shown that thyroid breadth has a correlation of ·1528 ± ·0293 in boys and ·2141 ± ·0265 in girls with height after age correction. This suffices to show that, as a matter of fact, height is unconsciously either not at all or only partially corrected for in judging enlargement by the category method, and to what degree it is allowed for it is impossible to say. The usefulness of any method of categories based on visual judgment is therefore definitely limited by the following considerations: (i) comparison between results from different observers using, however carefully, the same definition proves the presence of a very large personal equation, as will be shown in another paper†; (ii) unless an observer is working continuously he cannot be sure that he is himself maintaining the same standards after an interval of even a few months; (iii) it is impossible to make adequate correction for height. In the present research the method proved invaluable for a special purpose, but for future work it cannot be recommended except for the purposes of a *rapid and continuous survey to be made by a single worker.*

(c) *Estimation of thyroid size by circumference or diameter of the neck.* These measurements were made on girls only. Those measured in Cheshire were from Elementary Schools for the most part whilst those measured in London were taken from Central Schools. Since the second group were perhaps selected physically to some extent the effect of age on the neck circumference and diameter has been determined separately. For the purposes of obtaining curves for age correction only girls classified into Category 01 could be used and the means for

* By the biserial method. † See Reference, No. 9.

age-groups of these girls were fitted with straight lines over the limited range of ages except in the case of neck circumference in Cheshire girls, where a parabola gave a better fit. The curve formulae and smoothed means are shown in Table II.

TABLE II.

| | Age in years exactly | | | | | |
|---|---|---|---|---|---|---|
| | *11* | *12* | *13* | *14* | *15* | *16* |
| *Neck Diameter* (cm.) | | | | | | |
| 151 Cheshire girls ($d = 4{\cdot}254 + {\cdot}3481a$) ... ... | 8·08 | 8·43 | 8·78 | 9·13 | 9·47 | — |
| 68 London girls ($d = 5{\cdot}887 + {\cdot}2349a$) ... ... | — | — | 8·94 | 9·18 | 9·41 | 9·65 |
| *Neck Circumference* (cm.) | | | | | | |
| 151 Cheshire girls ($c = 6{\cdot}192 + 2{\cdot}955a - {\cdot}08815a^2$) | 28·0 | 29·0 | 29·7 | 30·3 | 30·7 | — |
| 68 London girls ($c = 23{\cdot}092 + {\cdot}6283a$) ... ... | — | — | 31·3 | 31·9 | 32·5 | 33·1 |

The mean heights, weights, neck and thyroid measurements of these two groups of "normal" girls, all corrected to age 13 by means of their separate growth curves (see Sections 5, 6), were as follows:

TABLE III.

| | Height | Weight | Neck Diameter | Neck Circumference | Thyroid Breadth |
|---|---|---|---|---|---|
| Cheshire girls ... | 146·8 ± ·4 | 37·7 ± ·3 | 8·78 ± ·03 | 29·71 ± ·09 | 3·42 ± ·03 |
| London girls ... | 148·0 ± ·6 | 38·5 ± ·5 | 8·94 ± ·04 | 31·26 ± ·14 | 3·28 ± ·04 |

Although the means were slightly higher for the London girls the differences were insignificant for height and weight and barely significant for neck diameter and thyroid measurements, but for neck circumference the difference of 1·55 ± ·16 was almost ten times its probable error. Moreover the ratios of circumference to "diameter" are respectively 3·38 and 3·49, which differ to the extent of 3 %. The London measurements were carried out at an interval of some months from the Cheshire series, and I believe these differences to be accounted for by the impossibility of maintaining a constant technique in the measurement of neck circumference, where so much depends on the levels at which the tape is kept. This experience strengthened a view I already held that neck circumference is entirely useless as a measure of thyroid enlargement, except perhaps to note the progress of a large goitre in the same individual from time to time by the same observer. This is further confirmed by the following correlation coefficients which have been calculated for the whole series of 540 girls after age correction.

TABLE IV.

| | | Neck Diameter | Neck Circumference |
|---|---|---|---|
| With thyroid breadth | Category 01 only ... | ·2220 ± ·0475 | ·2989 ± ·0455 |
| | Categories 2—3 only ... | ·6536 ± ·0336 | ·5378 ± ·0450 |
| | Complete sample ... | ·4279 ± ·0246 | ·2951 ± ·0275 |
| With height, Category 01 only ... ... ... | | ·5168 ± ·0368 | ·5607 ± ·0346 |

If the thyroid and neck measurements be corrected for height by the method described in Section 5 (*e*) the resulting coefficients with thyroid breadth are:

TABLE V.

| | Neck Diameter | Neck Circumference |
|---|---|---|
| Category 01 only ... | $r = ·0876$ | $r = ·1714$ |
| Categories 2—3 only ... | $r = ·6557$ | $r = ·5393$ |
| Complete sample ... | $r = ·4151$ | $r = ·2740$ |

Of the two neck measurements the antero-posterior diameter gives the better index of thyroid size, but the magnitude of these coefficients is such that in dealing with any but large goitres, neither measurement is of much value for this purpose, being inferior to visual estimation when used by a single observer (see Table VIII).

(*d*) *Actual measurements of the thyroid gland.* The instrument used consisted of one fixed and one sliding arm, each carrying ivory points, the distance between the points being indicated on a scale on the rod carrying the sliding arm, the latter having a vernier attachment. The boundaries of the gland having been defined by palpation with the thumb and forefinger of the left hand without undue pressure, the points of the instrument were brought into apposition with the boundaries and the distance between them read off. Any accurate pair of callipers would suffice for such measurement.

The maximum breadth, and the height of each lobe, were thus measured. Hunziker's surface area measure was also determined by multiplying the breadth by the arithmetic mean of the heights of the lobes; usually the heights of the lobes did not differ*. In my experience there was little difficulty in measuring breadth, but often considerable difficulty in defining the upper limits of the lobes with certainty. It seemed doubtful therefore whether any advantage was gained by introducing the additional vertical measurements over the simple measurement

* In 62 girls the right lobe was larger than the left, whereas in only nine cases was the left larger; there is some tendency for the right to enlarge first.

of breadth alone. The latter has the advantage of being easier to deal with in computation and undoubtedly is subject to a smaller personal equation than the area measurement. It is possible to decide which is the better measure to use from an examination of the distributions of the same children on both scales and their relation to age and visual estimates of size.

TABLE VI.

*Mean Thyroid Measurements on Girls.*

| | Age | Thyroid Breadth; cm. Smoothed Values | | | Thyroid Area; sq. cm. Smoothed Values | | |
|---|---|---|---|---|---|---|---|
| | | Mean | S.D. | Coefficient of Variation | Mean | S.D. | Coefficient of Variation |
| Normal Thyroids, Category 01 | *10* | 3·29 | ·400 | 12·2 | 7·50 | 1·60 | 21·3 |
| | *11* | 3·29 | ·405 | 12·3 | 7·60 | 1·66 | 21·9 |
| | *12* | 3·30 | ·433 | 13·1 | 7·66 | 1·75 | 22·9 |
| | *13* | 3·30 | ·482 | 14·6 | 7·75 | 1·95 | 25·2 |
| | *14* | 3·31 | ·508 | 15·4 | 8·20 | 3·20 | 39·0 |
| | *15* | 3·92 | ·517 | 13·2 | 12·00 | 3·70 | 30·8 |
| | *16* | 4·03 | ·518 | 12·8 | 13·10 | 2·90 | 22·1 |
| Slightly Enlarged Thyroids, Category 2 | *10* | 4·15 | ·59 | 14·2 | 11·58 | 3·50 | 30·2 |
| | *11* | 4·18 | ·61 | 14·6 | 13·44 | 4·10 | 29·7 |
| | *12* | 4·47 | ·73 | 16·3 | 15·09 | 5·00 | 33·1 |
| | *13* | 4·71 | ·93 | 19·8 | 16·51 | 5·80 | 35·1 |
| | *14* | 4·89 | ·93 | 19·0 | 17·71 | 5·82 | 32·9 |
| | *15* | 4·96 | ·78 | 15·7 | 18·68 | 5·70 | 30·5 |
| | *16* | 5·00 | ·62 | 12·4 | 19·44 | 5·50 | 28·3 |
| Definite Goitres, Category 3 | *10* | 7·12 | ·72 | 10·1 | 36·37 | 8·79 | 24·2 |
| | *11* | 7·12 | ·76 | 10·7 | 36·00 | 8·79 | 24·4 |
| | *12* | 7·12 | ·90 | 12·6 | 35·62 | 8·79 | 24·7 |
| | *13* | 7·09 | 1·10 | 15·5 | 35·24 | 8·79 | 24·9 |
| | *14* | 6·96 | 1·17 | 16·8 | 34·87 | 8·79 | 25·2 |
| | *15* | 6·90 | 1·03 | 14·9 | 34·89 | 8·79 | 25·5 |
| | *16* | 6·90 | ·82 | 11·9 | 34·12 | 8·79 | 25·7 |

The effect of age on thyroid growth is described in Section (4); it will suffice here to tabulate the smoothed means and standard deviations and the coefficients of variation at each age for comparison (Table VI)*. It appears that whereas the coefficient of variation of thyroid breadth averages about 15, that of thyroid area averages about 27; for both measures it has a maximal value about ages 13—14. Theoretically the coefficient of variation of the *square of the breadth* would be 29†.

* The data from which these are derived are set out in Table XV.

† [If $V_{x^2}$ be the coefficient of variation of $x^2$ and $V_x$ of $x$, then

$$V_{x^2} = 2V_x \sqrt{1+\tfrac{1}{2}(V_x/100)^2}/(1+(V_x/100)^2) = 2V_x \text{ approximately.}$$ E.S.P.]

The relations of the two measures with Category are shown in Table VII after correcting to age 12.

TABLE VII.

*Thyroid Breadth, corrected to age* 12.

Central values in cm.

| | | 2·5 | 3·0 | 3·5 | 4·0 | 4·5 | 5·0 | 5·5 | 6·0 | 6·5 | 7·0 | 7·5 | 8·0 | 8·5 | 9·0 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Boys.* | Category 01 ... | 17 | 82 | 144 | 65 | 2 | — | — | — | — | — | — | — | — | — | 310 |
| | „ 2 ... | — | — | 3 | 61 | 67 | 40 | 10 | 5 | 1 | — | — | — | — | — | 187 |
| | „ 3 ... | — | — | — | — | — | — | — | — | 4 | 1 | 1 | 2 | — | 1 | 9 |
| *Girls.* | Category 01 ... | 28 | 68 | 91 | 29 | 5 | — | — | — | — | — | — | — | — | — | 221 |
| | „ 2 ... | — | 2 | 22 | 67 | 65 | 42 | 20 | 20 | 9 | 2 | 1 | 1 | — | — | 251 |
| | „ 3 ... | — | — | — | — | 1 | 3 | 5 | 11 | 3 | 11 | 17 | 6 | 8 | 3 | 68 |

*Thyroid Area, corrected to age* 12.

Central values in sq. cm.

| | | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Girls.* | Category 01 ... | 102 | 110 | 9 | — | — | — | — | — | — | — | — | — | — | 221 |
| | „ 2 ... | 3 | 70 | 96 | 44 | 26 | 9 | 2 | 1 | — | — | — | — | — | 251 |
| | „ 3 ... | — | — | 3 | 4 | 10 | 11 | 13 | 14 | 6 | 2 | 4 | — | 1 | 68 |

The resulting correlation coefficients* computed by the triserial method, obtaining the means of the general categories from the probability integral†, were as follows:

TABLE VIII.

| | Boys | Girls |
|---|---|---|
| Thyroid Breadth, age corrected, with Category: | | |
| With class index correction $r=$ | ·8182 ± ·0099 | ·9118 ± ·0049 |
| Without „ „ $r=$ | ·6746 ± ·0163 | ·6825 ± ·0155 |
| Thyroid Area, age corrected, with Category: | | |
| With class index correction $r=$ | — | ·9040 ± ·0053 |
| Without „ „ $r=$ | — | ·6746 ± ·0157 |

* In the case of the girls the actual numbers in Category 01 were raised to bear the correct ratio to the other categories before computing $r$, since the 221 who were measured represented a random sample of girls in this category in the schools visited; this is explained in Section (5).

† An alternative method was used for girls by finding the slope of the line fitted by least squares to the three means and comparing with the regression line; this gave $r = ·9050$ for thyroid breadth with Category. We only need to determine the order of magnitude of these coefficients; owing to heterogeneity their values are somewhat dependent on the prevalence of goitre in the district sampled.

The general conclusion is that thyroid "area" offers no advantages over thyroid breadth. The two variables are very highly correlated, and the latter being very simple to measure is recommended as the best index of size to use in biometric work, or for comparative purposes.

### (3) FORM OF FREQUENCY DISTRIBUTION OF THYROID SIZE AMONGST CHILDREN AGED 12.

This must obviously be different in a goitre district to a non-goitre district, and the manner in which it changes as goitre increases ought to throw some light on the relation between the so-called "physiological" enlargements and goitres. In order to attack the problem directly and completely it would be necessary for a single observer to measure the thyroids of, say, 500 children of each sex in a dozen districts of differing goitre prevalence, a large undertaking which was not practicable. For the present an approximate method was used which, though open to objections, was believed to be sufficiently accurate for the immediate problems it was designed to meet. The general method of attack for all the purposes in view was as follows:

($\alpha$) 440 girls aged 10—15 in schools in the neighbourhoods of Runcorn, Stockton Heath, Lymm, Northwich and Knutsford in Cheshire, where goitre incidence is considerable, and at Altrincham where it is lower, were measured as regards the thyroid and physical and mental factors, and in addition 100 girls in London County Council schools. In the Cheshire group of schools about 15 % of the girls had thyroid enlargement according to my own standard, and therefore since the time available was limited all girls showing any enlargement (Categories 2 and 3) in the school visited were first measured and a random sample was then taken from the normals (Category 01) who remained as time permitted. No conscious selection was exercised within the limits of each of these groups, and it was therefore possible to work out mathematical expressions for the frequency distributions on a scale of thyroid breadth of each of the categories of girls separately.

($\beta$) In the course of an attempt to assess the extent of the personal equation of different School Medical Officers I was able to rapidly survey unselected samples of 150 boys and 150 girls in ten different districts of Somersetshire, Devonshire and Surrey and to record the numbers of each age 11, 12 and 13 who came within the three categories, using the same standards as before. The complete details of this survey will be published elsewhere (9).

($\gamma$) By making the approximate assumption that within the boundaries of each category the distribution of thyroid sizes would not differ seriously from district to district it was possible, by combining the three frequency distributions ascertained in ($\alpha$) in the proportions observed in the three categories in a locality as in ($\beta$), to roughly construct the distribution of thyroid sizes for that locality.

($\delta$) 533 boys aged 11—14 were measured in rural schools in the neighbourhood of Dartmoor and the S. Devonshire coast, and here the goitre incidence was so high

that it was practicable to measure *every* boy of these ages in the schools visited. Separate frequency distributions for the categories were fitted with curves as for the girls, and the complete distribution resulting from the approximate method ($\gamma$) compared with the total observed distribution in a sample area.

($\epsilon$) A similar check on the validity of the approximate method for girls was secured later by rapidly measuring the thyroid breadths alone of a complete school; this was carried out in Stockport by the measurement of 372 girls.

It will suffice here to reduce to a mathematical form the frequency distributions in the three categories on a scale of thyroid breadth, and to give one or two examples of the estimated composite curves. For obvious reasons it is difficult in rural schools to obtain large enough numbers by confining attention to a single year of age, hence the method used has been to measure children aged 11, 12 and 13 last birthday, calculate separate moments for each age-group and combine them, giving equal weight to each of the three ages; the resultant moments and curves are taken as applicable to the age-group "12."

*Category 01.* (Normal thyroids according to my judgment.)

*Boys.* The frequencies and resulting constants are given in Table IX.

TABLE IX.

| Thyroid breadth. Central values | Age last Birthday | | | Percentage frequency. Ages equally weighted |
|---|---|---|---|---|
| | *11* | *12* | *13* | |
| 2·37 cm. | 1 | — | — | 6·40 |
| 2·62 „ | 8 | 3 | 7 | |
| 2·87 „ | 8 | 10 | 9 | 8·93 |
| 3·12 „ | 20 | 19 | 19 | 19·25 |
| 3·37 „ | 19 | 26 | 18 | 20·76 |
| 3·62 „ | 21 | 26 | 24 | 23·48 |
| 3·87 „ | 13 | 18 | 13 | 14·50 |
| 4·12 „ | 7 | 5 | 7 | 6·68 |
| 4·37 „ | — | — | 1 | |
| Totals | 97 | 107 | 98 | 100·00 |

Taking 3·37 group as origin and the class interval as unit, for age 11, $\nu_1' = \cdot04124$, $\nu_2' = 2\cdot84536$; age 12, $\nu_1' = \cdot27103$, $\nu_2' = 2\cdot14019$; age 13, $\nu_1' = \cdot17347$, $\nu_2' = 2\cdot78571$. Combining the ages equally $\nu_1' = \cdot16258$ and transferring to the mean as origin and applying Sheppard's correction, $\mu_2 = 2\cdot48066$. Hence mean breadth = 3·4106 cm., $\sigma = \cdot39375$, and fitting a normal curve this gives for unit total frequency

$$y_1 = 1\cdot01319e^{-\frac{(x-3\cdot4106)^2}{\cdot31008}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1\,a),$$

where $x$ is the thyroid breadth in cm.

This curve when tested for goodness of fit with the frequencies in the last column of Table IX raised to the total number of boys measured (302) gives for seven groups $\chi^2 = 5{\cdot}66$ and $P = {\cdot}46$, so the fit is satisfactory as shown in Figure 1 *a*.

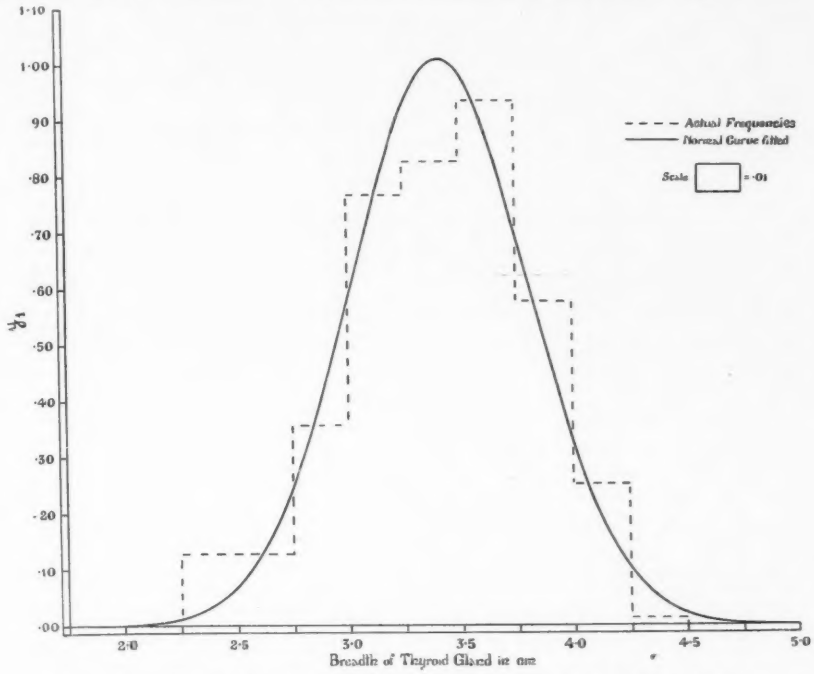



Fig. 1 *a*. Frequency distributions of Thyroid Breadth in Boys aged 11—13 whose thyroids were considered by author to be normal or subnormal (Category 01).

*Girls.* Different class intervals were used owing to the difficulty of disposing of the group of impalpable thyroids. See Table X.

Taking 3·495 group as origin and the class interval as unit and proceeding as above, for ages 11—13 we have $\nu_1' = -{\cdot}35462$, and transferring to the mean as origin and applying Sheppard's correction, $\mu_2 = {\cdot}65453$. Hence mean breadth = 3·3177 cm., $\sigma = {\cdot}40452$ cm., and fitting a normal curve this gives for unit total frequency

$$y_1 = {\cdot}98621e^{-\frac{(x-3{\cdot}3177)^2}{{\cdot}32726}} \qquad \text{.....(1 } b\text{).}$$

Testing for goodness of fit* with smaller class intervals as shown in Figure 1 *b*, for six groups $\chi^2 = 5{\cdot}55$ and $P = {\cdot}36$, and for five groups $\chi^2 = 3{\cdot}54$, $P = {\cdot}48$. The

* The total frequency 161 was used. This total is perhaps not strictly applicable, owing to the method of equally weighting the age-groups. Hence this and some of the other values of $\chi^2$ calculated in this section must be treated as approximations only.

TABLE X.

| Thyroid breadth. Central values | Age last Birthday | | | Percentage frequency. Ages equally weighted |
|---|---|---|---|---|
| | *11* | *12* | *13* | |
| 2·495 cm. | 4 | 2 | 14 | 10·95 |
| 2·995 „ | 10 | 7 | 28 | 27·20 |
| 3·495 „ | 17 | 16 | 41 | 48·55 |
| 3·995 „ | 3 | 5 | 13 | 12·96 |
| 4·495 „ | — | — | 1 | ·34 |
| Totals | 34 | 30 | 97 | 100·00 |

misfit at the start of this curve can be explained by the disposal of all the impalpable thyroids into the first group whereas some of them doubtless belonged to the second.

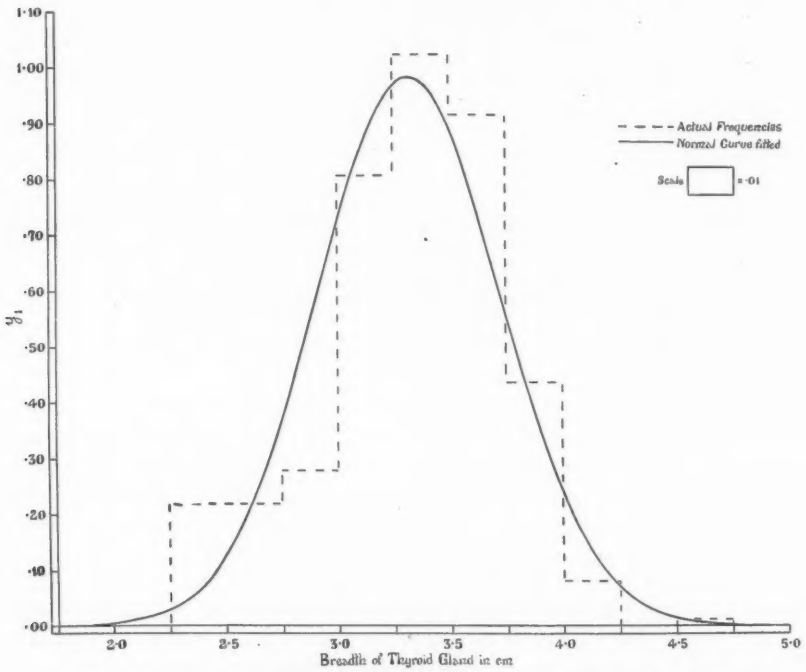



Fig. 1 *b*. Frequency distributions of Thyroid Breadth in Girls aged 11—13 whose thyroids were considered by author to be normal or subnormal (Category 01).

*Category 2.* (Thyroids slightly or moderately enlarged.) See Table XI.

*Boys.* Combining the three ages equally the resulting moments without applying Sheppard's correction are, from 4·495 as origin $\nu_1' = \cdot 12231$, and referred to

TABLE XI.

| Thyroid breadth Central values | Boys | | | Girls | | | Percentage frequencies. Ages equally weighted | |
|---|---|---|---|---|---|---|---|---|
| | Age last Birthday | | | Age last Birthday | | | | |
| | *11* | *12* | *13* | *11* | *12* | *13* | Boys | Girls |
| 3·495 cm. | — | 2 | 1 | 7 | 7 | 6 | 1·45 | 11·30 |
| 3·995 „ | 17 | 27 | 18 | 16 | 12 | 27 | 31·35 | 28·25 |
| 4·495 „ | 23 | 19 | 29 | 9 | 20 | 26 | 36·30 | 26·35 |
| 4·995 „ | 14 | 13 | 12 | 4 | 13 | 12 | 20·10 | 13·95 |
| 5·495 „ | 3 | 7 | 4 | 3 | 4 | 10 | 7·00 | 7·85 |
| 5·995 „ | — | 2 | 4 | 2 | 7 | 9 | 2·85 | 8·25 |
| 6·495 „ | — | — | 2 | — | 2 | 6 | ·95 | 4·05 |
| 6·995 „ | — | — | — | — | — | 2 | — | |
| 7·495 „ | — | — | — | — | — | 1 | — | |
| Totals | 57 | 70 | 70 | 41 | 65 | 99 | 100·00 | 100·00 |

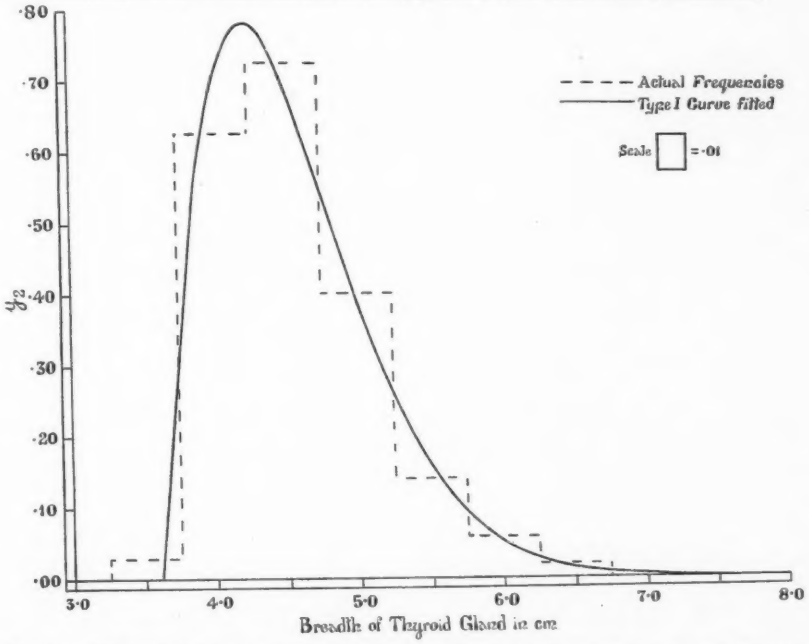



Fig. 2 *a*. Frequency distribution of Thyroid Breadth in Devonshire Boys aged 11—13 whose thyroids were considered by author to exhibit slight or moderate enlargement (Category 2).

the mean $\nu_2 = 1{\cdot}24603$, $\nu_3 = 1{\cdot}25426$, $\nu_4 = 5{\cdot}88869$, giving $\beta_1 = {\cdot}81319$, $\beta_2 = 3{\cdot}7928$. Hence a Type I Pearsonian skew curve can be fitted. Proceeding to fit such a curve with the above constants without attempting to apply correction for abruptness we have in the usual notation $r = 13{\cdot}9102$, $b = 5{\cdot}8830$ cm., $a_2 = 5{\cdot}2786$ cm., $a_1 = {\cdot}6044$ cm., $m_2 = 10{\cdot}6886$, $m_1 = 1{\cdot}2216$. Hence the mode is at 4·2200 cm., the mean at 4·5561 cm., the curve starts at 3·6156 cm. and ends at 9·4986 cm. and has the formula, for unit total frequency,

$$y_2 = {\cdot}78140 \left(\frac{x - 3{\cdot}6156}{{\cdot}6044}\right)^{1{\cdot}2216} \left(\frac{9{\cdot}4986 - x}{5{\cdot}2786}\right)^{10{\cdot}6886} \quad \ldots\ldots\ldots\ldots (2\,a).$$

Testing for goodness of fit as before, for seven groups $\chi^2 = 1{\cdot}02$, $P = {\cdot}98$, so the fit is excellent as shown by Figure 2 *a*.

*Girls.* Combining the ages equally the resulting moments without Sheppard's corrections and without abruptness corrections are, from 4·495 as origin $\nu_1' = {\cdot}21128$, and referred to the mean $\nu_2 = 2{\cdot}58080$, $\nu_3 = 3{\cdot}33804$, $\nu_4 = 21{\cdot}6968$. Thus without high contact or abruptness corrections $\beta_1 = {\cdot}64822$, $\beta_2 = 3{\cdot}25752$ and a Type I curve can be fitted as for the boys. The constants of this are $r = 6{\cdot}7541$, $b = 5{\cdot}2938$ cm., $a_2 = 4{\cdot}6076$ cm., $a_1 = {\cdot}6362$ cm., $m_2 = 4{\cdot}1828$, $m_1 = {\cdot}5713$. Hence the mode is at 4·0052 cm., the mean at 4·6006 cm., the curve starts at 3·3691 cm. and ends at 8·6129 cm., and its formula for unit total frequency is

$$y_2 = {\cdot}53367 \left(\frac{x - 3{\cdot}3691}{{\cdot}6362}\right)^{{\cdot}5713} \left(\frac{8{\cdot}6129 - x}{4{\cdot}6076}\right)^{4{\cdot}1828} \quad \ldots\ldots\ldots\ldots\ldots (2\,b).$$

Testing for goodness of fit as before, for seven groups $\chi^2 = 6{\cdot}95$, $P = {\cdot}33$, and for six groups $\chi^2 = 4{\cdot}12$, $P = {\cdot}53$, which is satisfactory, as shown in Figure 2 *b*.

*Category 3.* (Large goitres.) See Table XII.

TABLE XII.

| Thyroid breadth. Central values | Boys | Girls | |
|---|---|---|---|
| | Ages 11—14 | Ages 11—14 | |
| 4·495 | — | 2 | 2 |
| 4·995 | — | 3 | 6 |
| 5·495 | — | 3 | |
| 5·995 | — | 12 | 15 |
| 6·495 | 3 | 3 | |
| 6·995 | 2 | 12 | 29 |
| 7·495 | 1 | 17 | |
| 7·995 | 1 | 6 | 14 |
| 8·495 | — | 8 | |
| 8·995 | 1 | 3 | 3 |
| Totals | 8 | 69 | |

The means and standard deviations for separate ages from 11 to 14 showed little or no change with age in this group so they have been combined. In boys the number in this category was naturally too small to give any indication of the form of frequency distribution, but since the corresponding distribution for girls can be satisfactorily represented by a normal curve, this type of curve has also been fitted to the scanty data for boys (8 only!) for the sake of uniformity in the two sexes, using the actual measurements to calculate the moments. Thyroids of this size are so infrequent in boys that the precise method of their disposal along the tail of the compound curve makes little difference in any calculations.

*Boys.* Mean breadth 7·2612 cm. $\sigma = \cdot 7754$ cm.

$$y_3 = \cdot 51451 e^{-\frac{(x-7\cdot 2612)^2}{1\cdot 2024}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3\,a).$$

*Girls.* Mean breadth 7·0457 cm. $\sigma = 1\cdot 0943$ cm.

$$y_3 = \cdot 36455 e^{-\frac{(x-7\cdot 0457)^2}{2\cdot 3952}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3\,b).$$

This last curve tested for goodness of fit gives for six groups $\chi^2 = 2\cdot 80$, $P = \cdot 73$.

It was found impossible after many attempts to fit any simpler form of mathematical curve satisfactorily to the combined data for Categories 2 and 3; the separate treatment of these two categories must not be taken to imply any difference in nature or causation between them—in fact I can find no evidence to

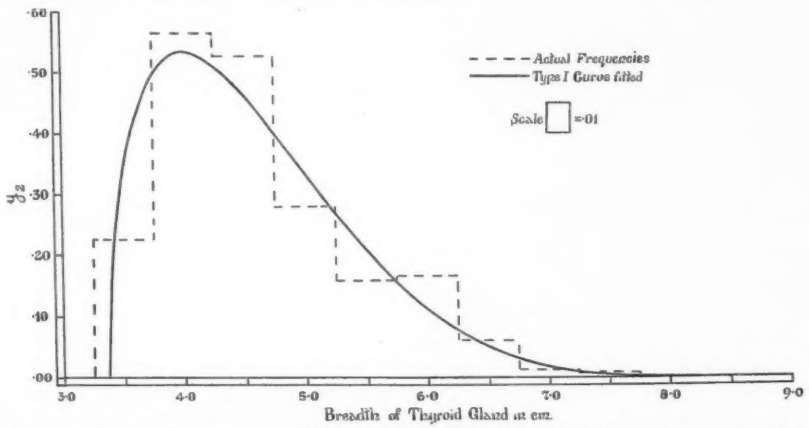



Fig. 2 *b*. Frequency distribution of Thyroid Breadth in Cheshire Girls aged 11—13 whose thyroids were considered by author to exhibit slight or moderate enlargement (Category 2).

suggest that the distinction between small "physiological" enlargements and the larger goitres as usually met with in this country is more than one of degree—the method here used is merely an expedient to obtain a nearer approximation to the form of the total frequency distribution in mathematical form. As a matter

of fact a collection of the available evidence bearing on the question, which will be published elsewhere (9), leads to the conclusion that in comparing districts of differing goitre prevalence all degrees of enlargement above the average estimate of normality in children aged 12 are found to be present in approximately constant proportions within the limits of sampling errors; that is to say each large goitre is attended by a fairly constant proportion of small goitres. For my own delimitation of Categories 2 and 3 it appears that the relative proportions in which they occur would on this supposition approximate to 9 to 2 in girls and 49 to 1 in boys if large enough numbers were examined.

If $\gamma_1$, $\gamma_2$, $\gamma_3$ be the proportions per unit total frequency in the Categories 01, 2, 3 respectively met with in a given district, an approximation to the complete frequency distribution in that district is arrived at by

$$y = \gamma_1 y_1 + \gamma_2 y_2 + \gamma_3 y_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where $y_1$, $y_2$, $y_3$ are given by equations (1), (2), (3). In Figure 3 this has been applied as an example to a fairly homogeneous area bordering the S. Devonshire

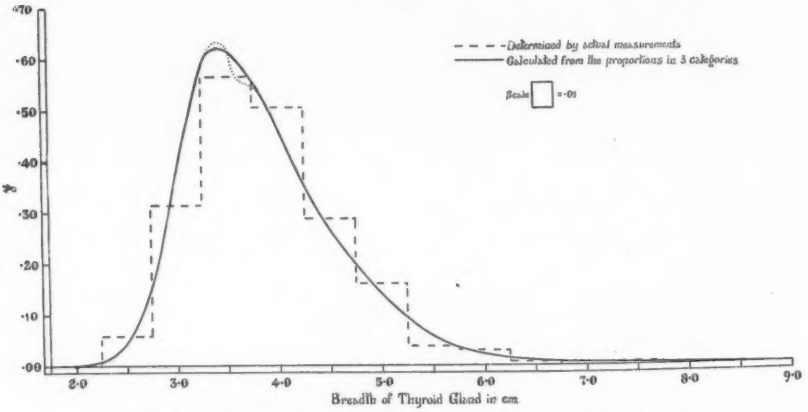



Fig. 3. Frequency distributions of thyroid breadth in 249 boys aged 11—13 at schools near the S. Devonshire coast.

coast in which an examination of 249 unselected boys aged 11—13 gave $\gamma_1 = \cdot 624$, $\gamma_2 = \cdot 365$, $\gamma_3 = \cdot 011$. The calculated curve* is drawn for comparison with the observed frequencies, which were:

| Thyroid breadth. Central values | 2·5 | 3 | 3·5 | 4 | 4·5 | 5 | 5·5 | 6 | 6·5 | 7 | 7·5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency per unit total | ·029 | ·157 | ·283 | ·253 | ·144 | ·092 | ·019 | ·015 | ·004 | — | ·004 |

* Slight smoothing is called for at the crest of the curve, but this has no appreciable effect on calculations for goitre rates below 50 %.

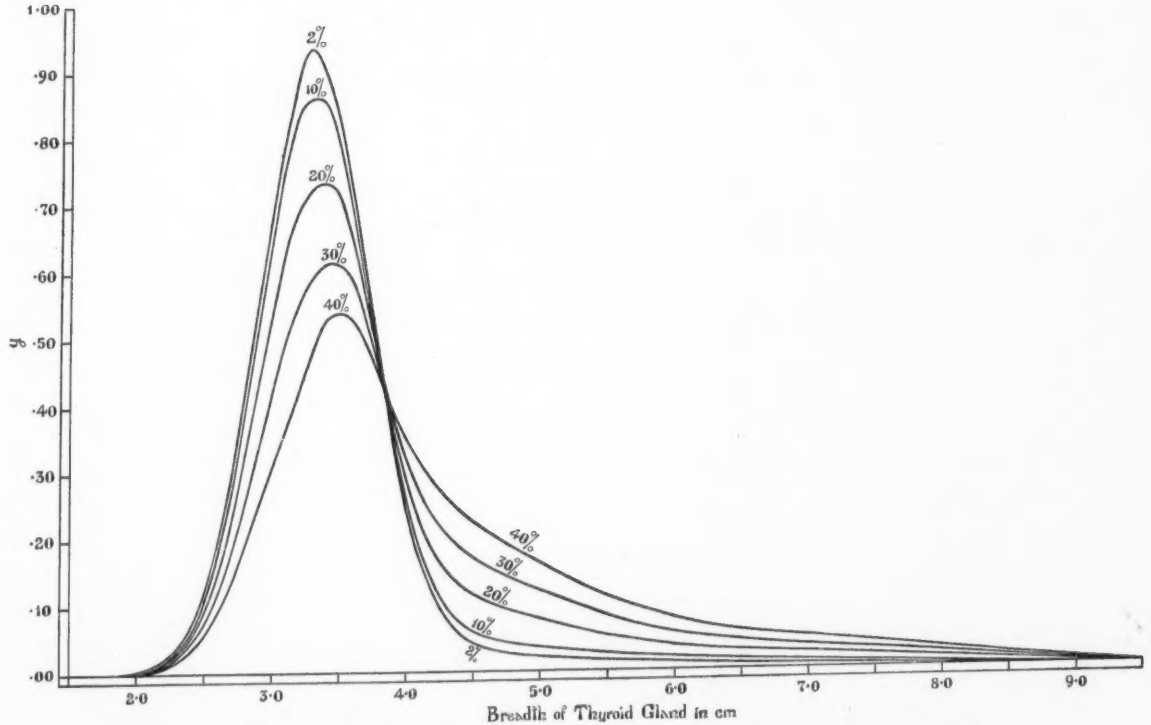



Fig. 4. Diagram constructed by the formulae of Section (3), to illustrate approximately the change in form of the frequency distribution of thyroid breadth in girls aged 11—13 with increasing prevalence of goitre (here defined as percentage having a thyroid breadth exceeding 42 mm.).

When tested for goodness of fit, raising the total to 249, with eight groups $\chi^2 = 4{\cdot}75$, $P = {\cdot}69$.

If only the total rate of thyroid enlargement, $g_0$, according to my own standard be known, $g_0 = \gamma_2 + \gamma_3$, $\gamma_1 = 1 - g_0$, and on the assumption of constancy in the ratio $\gamma_2 : \gamma_3$ as estimated above the complete distributions would be

for boys $y = (1 - g_0)\, y_1 + {\cdot}98\ g_0 y_2 + {\cdot}02\ g_0 y_3$ .....(5 *a*),

for girls $y = (1 - g_0)\, y_1 + {\cdot}818 g_0 y_2 + {\cdot}182 g_0 y_3$ .....(5 *b*).

As a result of a standardizing survey which I carried out in conjunction with 19 School Medical Officers in Devonshire, Somersetshire and Surrey I found that $g_0$ might be related to the average observer's estimate, $\bar{g}_r$, of the rate of thyroid enlargement in the same sample of children aged 11—13 by the formulae

for boys $g_0 = 1{\cdot}1047 \bar{g}_r + {\cdot}0327$ .....(6 *a*),

for girls $g_0 = 1{\cdot}0357 \bar{g}_r + {\cdot}0521$ .....(6 *b*),

or if it was desired to express the rate more usefully in terms of a suitable limiting thyroid breadth such as 42 mm., if $g_s$ be the proportion of boys or girls aged 11—13 with thyroid breadth exceeding the standard 42 mm., then

for boys $g_s = {\cdot}680 g_0 + {\cdot}0225$ .....(7 *a*),

for girls $g_s = {\cdot}673 g_0 + {\cdot}0147$ .....(7 *b*).

On the basis of these last equations and the preceding assumptions the curves in Figure 4 have been drawn to show the kind of way in which the distribution of sizes changes with increasing goitre prevalence. As a check on their validity I have used the measurements made on unselected Stockport girls after these curves had been constructed; these are given in Table XVII. The mean percentage of girls, aged 11, 12, 13, whose thyroid breadths exceeded 42 mm. was 19·12. Hence from equation (7 *b*) $g_0 = {\cdot}262$ and from equation (5 *b*) the resulting curve is given by $y = {\cdot}738 y_1 + {\cdot}214 y_2 + {\cdot}048 y_3$, which is drawn in Figure 6 for comparison with the observed frequencies in girls aged $11\frac{1}{2}$ to $13\frac{1}{2}$. The test for goodness of fit to this series of 131 girls (see Table XIX) gives for six groups $\chi^2 = 4{\cdot}68$, $P = {\cdot}46$.

The above method of treatment will no doubt seem clumsy or even indefensible to some biometricians but the nature of the problem presents unique features which have not as yet been explored by exact methods, and the present exploration of it is merely tentative to prepare the ground for further work.

### (4) Relation of thyroid size to age.

The construction of a growth curve for the thyroid gland is a matter of some difficulty owing to (i) rapid changes in goitre incidence with age about the period of puberty, (ii) the fact that practically all measurements of the gland have been made in areas where goitre is prevalent, (iii) uncertainty as to what limits should be set to "normality" in the size of the gland at each age, (iv) difficulty in measuring the gland at ages before nine owing to its smallness. It is first necessary to be clear as to the *changes in goitre incidence with age*. In Table XIII data from various sources have been collected and made comparable by expressing the frequency at each age of "goitre" or "thyroid enlargement" in terms of the

TABLE XIII. *Incidence of Thyroid Enlargement at various ages expressed in terms of the frequency at age* 12 *measured by the same observer in the same locality.*

| Locality | Observer | Criterion | Age last Birthday | | | | | | | | | | | | Total examined |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | *5—6* | *7* | *8* | *9* | *10* | *11* | *12* | *13* | *14* | *15* | *16* | *17—19* | *20—25* | |
| *Girls* | | | | | | | | | | | | | | | | |
| Bern ... ... ... | Kerzmann and de Quervain (3) | Categories III—IV | — | ·537 | ·495 | ·766 | ·756 | ·832 | 1 | 1·066 | 1·057 | 1·178 | 1·171 | — | — | 4038 |
| Neuchatel, etc. ... ... | | | — | ·528 | ·553 | ·765 | ·796 | ·913 | 1 | 1·228 | 1·344 | — | — | — | — | 981 |
| Lancashire, 1924 ... ... | Holroyd ... | Visibly enlarged thyroids | ·086 | ·228 | ·300 | ·471 | ·612 | ·632 | 1 | ·938 | 1·125 | — | — | — | — | 3346 |
| Derbyshire, Heanor ... | Turton ... | | ·738 | ·800 | ·890 | ·880 | ·738 | ·880 | 1 | 1·143 | 1·472 | 1·348 | 1·363 | 1·291 | 1·122 | 2047 |
| Cheshire, N.E., 1924 ... | A. V. Stocks ... | | — | — | ·192 | — | — | 1·0 | 1 | 1·231 | — | — | — | — | — | — |
| Stockport, 1926 ... ... | P. Stocks ... | | — | — | — | — | ·729 | ·838 | 1 | 1·045 | 1·059 | 1·099 | 1·229 | — | — | 362 |
| Somerset and Devon ... | P. Stocks ... | Cat. 2—3 | — | — | — | — | — | 1·008 | 1 | 1·159 | — | — | — | — | — | 1440 |
| Shropshire, 1924 ... ... | School Medical Officers | Visibly enlarged thyroids | ·110 | — | ·252 | — | — | — | 1 | — | — | — | — | — | — | 5610 |
| Gloucestershire, 1911—21 | | | ·097 | — | — | — | — | — | 1 | — | — | — | — | — | — | 40,932 |
| Yorkshire, E. Riding, 1924 | | | — | — | ·430 | — | — | — | 1 | 1·208 | — | — | — | — | — | 3044 |
| Isle of Wight, 1924 ... | | | — | — | — | — | — | — | 1 | 1·060 | — | — | — | — | — | 741 |
| Glossop, 1924—25 ... | Milligan ... | | — | — | ·057 | — | — | — | 1 | — | — | — | — | — | — | 403 |
| Cincinnati, U.S.A., 1924 ... | Olesen (6) ... | Enlargement of any degree | ·407 | ·496 | ·593 | ·673 | ·764 | ·837 | 1 | 1·074 | 1·082 | 1·028 | 1·109 | — | — | 21,009 |
| Minnesota, U.S.A., 1924 ... | „ (4) ... | | — | — | — | — | — | ·818 | 1 | 1·064 | 1·078 | 1·034 | 1·057 | ·989 | — | 2260 |
| Connecticut, U.S.A., 1925 | „ (5) ... | | — | — | — | — | — | ·809 | 1 | 1·220 | 1·212 | 1·295 | 1·334 | 1·375 | — | 6603 |
| Mean rate ... ... | — | — | ·288 | ·518 | ·418 | ·708 | ·732 | ·857 | 1 | 1·120 | 1·181 | 1·164 | 1·210 | 1·218 | 1·122 | — |
| *Boys* | | | | | | | | | | | | | | | | |
| Bern ... ... ... | Kerzmann and de Quervain (3) | Categories III—IV | — | ·579 | ·655 | ·869 | ·874 | ·922 | 1 | 1·041 | 1·227 | 1·327 | 1·388 | — | — | 3527 |
| Neuchatel, etc. (1) ... | | | — | ·318 | ·830 | ·630 | ·925 | ·857 | 1 | 1·101 | 1·040 | — | — | — | — | 1001 |
| Lancashire, 1924 ... ... | Holroyd ... | Visibly enlarged thyroids | ·210 | ·249 | ·712 | ·531 | ·783 | ·634 | 1 | ·992 | 1·048 | — | — | — | — | 3247 |
| Cheshire, N.E., 1924 ... | A. V. Stocks ... | | — | — | ·548 | — | — | ·809 | 1 | ·952 | — | — | — | — | — | — |
| Somerset and Devon ... | P. Stocks ... | Cat. 2—3 | — | — | — | — | — | ·944 | 1 | ·921 | — | — | — | — | — | 2016 |
| Shropshire, 1924 ... ... | School Medical Officers | Visibly enlarged thyroids | — | — | ·536 | — | — | — | 1 | — | — | — | — | — | — | 5534 |
| Gloucestershire, 1911—21 | | | ·300 | — | — | — | — | — | 1 | — | — | — | — | — | — | 41,131 |
| Yorkshire, E. Riding, 1924 | | | — | — | ·537 | — | — | — | 1 | ·774 | — | — | — | — | — | 3239 |
| Isle of Wight, 1924 ... | | | — | — | — | — | — | — | 1 | 1·368 | — | — | — | — | — | 754 |
| Glossop, 1924—25 ... | Milligan ... | | — | — | ·273 | — | — | — | 1 | — | — | — | — | — | — | 358 |
| Cincinnati, U.S.A., 1924 ... | Olesen (6) ... | Enlargement of any degree | ·505 | ·559 | ·653 | ·752 | ·825 | ·876 | 1 | 1·042 | ·943 | ·822 | ·737 | — | — | 21,299 |
| Minnesota, U.S.A., 1924 ... | „ (4) ... | | — | — | — | — | — | 1·023 | 1 | ·942 | 1·010 | ·725 | ·721 | ·508 | — | 1691 |
| Connecticut, U.S.A., 1925 | „ (5) ... | | — | — | — | — | — | 1·757 | 1 | 1·171 | 1·286 | ·914 | ·700 | ·500 | — | 5787 |
| Mean rate ... ... | — | — | ·338 | ·426 | ·593 | ·695 | ·852 | ·978 | 1 | 1·030 | 1·092 | ·947 | ·887 | ·504 | — | — |

frequency at age 12—13 measured by the same observer in the same locality. It must be admitted that these figures show great variation at the early ages even when so treated, but nevertheless the mean rates obtained by combining all the observations form a tolerably smooth series and have been fitted in Figure 5 with parabolae passing through unit rate at age 12·5 for boys and girls respectively. The equations of these curves are:

$$\text{Boys} \quad z = -1{\cdot}6924 + {\cdot}40065a - {\cdot}014821a^2 \quad \ldots\ldots\ldots\ldots\ldots(8\,a),$$

$$\text{Girls} \quad z = -1{\cdot}1855 + {\cdot}26754a - {\cdot}007416a^2 \quad \ldots\ldots\ldots\ldots\ldots(8\,b),$$

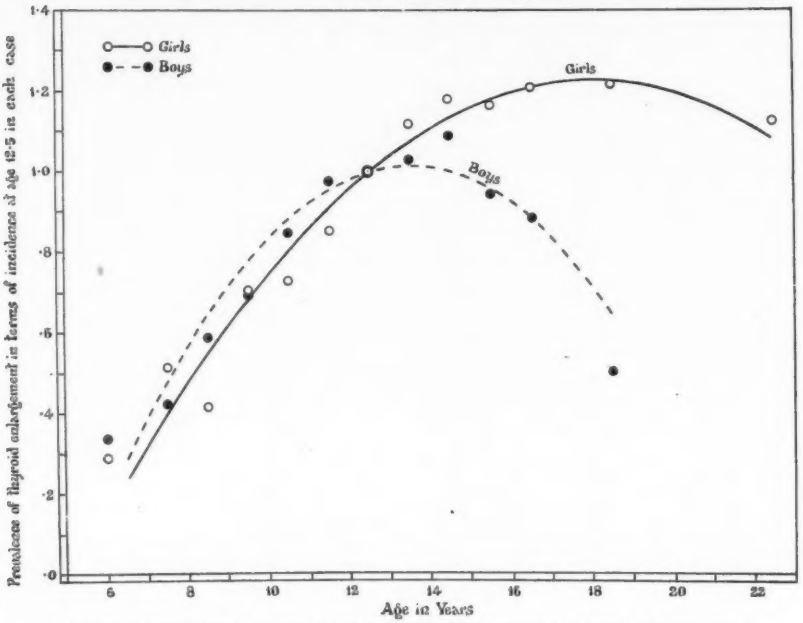



Fig. 5. Curves showing average effect of age on the frequency of thyroid enlargement.

where $z$ is the prevalence in terms of that at age $12\frac{1}{2}$ and $a$ is the age in years. The values of $z$ at ages from 9·5 to 16·5 are shown in Table XIV.

TABLE XIV.

| Age in Years | *9·5* | *10·5* | *11·5* | *12·5* | *13·5* | *14·5* | *15·5* | *16·5* |
|---|---|---|---|---|---|---|---|---|
| Boys ... | ·776 | ·881 | ·955 | 1 | 1·015 | 1·001 | ·957 | ·883 |
| Girls ... | ·687 | ·806 | ·911 | 1 | 1·075 | 1·135 | 1·180 | 1·210 |

After these curves had been drawn, their close similarity with a pair of age curves relating to children at Timaru given by Hercus, Benson and Carter in their paper on goitre in New Zealand (7) was noticed; their maxima however

TABLE XV. *Breadth of Thyroid Gland in* 581 *Girls, and* 645 *Boys, classified by Categories.*

| | Central Age | Category 01 | | | | | | | | Category 2 | | | | | | | | | | Category 3 | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | Thyroid Breadth* | | | | | | Totals | †Mean breadth, cm. | Thyroid Breadth* | | | | | | | | | Totals | †Mean breadth, cm. | Thyroid Breadth* | | | | | | | | | | Totals | †Mean breadth, cm. |
| | | 2·25– | 2·75– | 3·25– | 3·75– | 4·25– | 4·75– | | | 3·25– | 3·75– | 4·25– | 4·75– | 5·25– | 5·75– | 6·25– | 6·75– | 7·25– | 7·75– | | | 4·25– | 4·75– | 5·25– | 5·75– | 6·25– | 6·75– | 7·25– | 7·75– | 8·25– | 8·75– | | |
| Girls | 9·5‡ | 1 | 9 | 11 | 2 | — | — | 23 | 3·29 | 3 | 3 | 4 | — | — | — | — | — | — | — | 10 | 4·12 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 10·75 | — | 1 | — | 1 | — | — | 2 | 3·29 | — | 1 | — | 1 | 1 | — | — | — | — | — | 3 | 4·19 | — | — | — | — | 1 | — | — | 1 | — | — | 2 | 7·12 |
| | 11·25 | 4 | 3 | 11 | 2 | — | — | 20 | | 4 | 7 | 1 | 1 | 1 | — | — | — | — | — | 14 | | — | — | — | 1 | — | — | 1 | — | — | — | 2 | |
| | 11·75 | 7 | 6 | 1 | — | — | — | 14 | 3·30 | 3 | 9 | 8 | 3 | 2 | 2 | — | — | — | — | 27 | 4·47 | — | 1 | — | — | 1 | 1 | 1 | 1 | 1 | — | 7 | |
| | 12·25 | 1 | 4 | 8 | 1 | — | — | 14 | | 5 | 7 | 6 | 6 | 2 | 2 | — | — | — | — | 28 | | — | — | — | — | 1 | 1 | 1 | 1 | — | 9 | | |
| | 12·75 | 1 | 3 | 8 | 4 | — | — | 16 | | 2 | 5 | 14 | 7 | 2 | 5 | 2 | — | — | — | 37 | 4·71 | — | — | — | 1 | 1 | 2 | 2 | 2 | 1 | — | 9 | 7·07 |
| | 13·25 | 7 | 7 | 15 | 5 | 1 | — | 35 | 3·30 | 3 | 14 | 11 | 7 | 4 | 2 | 1 | 1 | — | — | 43 | | 1 | 1 | 1 | 3 | 1 | 4 | 5 | 2 | 1 | 2 | 21 | |
| | 13·75 | 7 | 21 | 26 | 8 | — | — | 62 | | 3 | 13 | 15 | 5 | 6 | 7 | 5 | 1 | 1 | — | 56 | 4·89 | — | 1 | 1 | 3 | — | 1 | 4 | 1 | 2 | — | 13 | |
| | 14·25 | 5 | 11 | 9 | 5 | 3 | — | 33 | 3·35 | — | 7 | 10 | 7 | 1 | 1 | 2 | — | — | 1 | 29 | | 1 | — | 1 | 1 | — | 3 | 3 | — | 3 | — | 12 | |
| | 14·75 | — | 2 | 4 | 3 | 1 | 1 | 11 | 3·80 | 1 | — | 3 | — | 3 | — | — | — | — | — | 7 | 4·97 | — | — | — | — | — | — | — | — | — | — | 2 | 6·90 |
| | 15·25 | — | — | 2 | 2 | — | 2 | 6 | | — | — | 1 | — | 1 | — | 1 | — | — | — | 3 | | — | — | — | — | — | 1 | — | — | — | — | 1 | |
| | 15·75 | — | — | 2 | 2 | 2 | — | 6 | 4·03 | — | 1 | 2 | — | 1 | 1 | — | — | — | — | 5 | | — | — | — | — | — | — | — | — | — | — | — | — |
| | 16·25 | — | — | — | — | 1 | — | 1 | | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — |
| | 20§ | — | — | 3 | 4 | 3 | 1 | 11 | 4·03 | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — | — |
| Boys | 10·75 | — | 7 | 4 | 1 | — | — | 12 | 3·22 | — | — | 2 | — | — | — | — | — | — | — | 2 | 4·49 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 11·25 | 4 | 9 | 18 | 9 | — | — | 40 | 3·41 | — | 9 | 13 | 5 | 2 | — | — | — | — | — | 29 | | — | — | — | — | — | — | — | — | — | — | 1 | |
| | 11·75 | 5 | 12 | 18 | 10 | — | — | 45 | | — | 8 | 8 | 9 | 1 | — | — | — | — | — | 26 | 4·51 | — | — | — | — | 1 | — | — | — | — | — | 1 | |
| | 12·25 | 2 | 14 | 26 | 17 | — | — | 59 | 3·43 | 1 | 15 | 9 | 5 | 3 | 2 | — | — | — | — | 35 | | — | — | — | — | 1 | — | — | — | — | — | 1 | 7·27 |
| | 12·75 | 1 | 15 | 26 | 6 | — | — | 48 | | 1 | 12 | 10 | 8 | 4 | — | — | — | — | — | 35 | 4·53 | — | — | — | — | — | — | — | — | — | — | — | |
| | 13·25 | 3 | 14 | 27 | 12 | 1 | — | 57 | 3·41 | 1 | 11 | 13 | 7 | 1 | — | — | — | — | — | 33 | | — | — | — | — | — | — | — | 1 | — | 1 | 2 | |
| | 13·75 | 4 | 14 | 15 | 8 | — | — | 41 | | — | 7 | 16 | 5 | 3 | 4 | 2 | — | — | — | 37 | 4·60 | — | — | — | — | 2 | 1 | 1 | — | — | — | 4 | |
| | 14·25 | 2 | 17 | 24 | 6 | 1 | — | 50 | 3·42 | — | 3 | 2 | 1 | — | — | — | — | — | — | 6 | | — | — | — | — | — | 1 | — | 1 | — | — | 2 | |
| | 14·75 | — | 6 | 16 | 6 | — | — | 28 | | — | 1 | 5 | 2 | 2 | — | — | — | — | — | 10 | | — | — | — | — | — | — | — | — | — | — | — | |
| | 15·25 | — | 3 | 9 | 1 | 1 | — | 14 | | — | 1 | — | — | — | 1 | — | — | — | — | 2 | 4·55 | — | — | — | — | — | — | — | — | — | — | — | |
| | 15·75 | — | 1 | 7 | 6 | — | — | 14 | 3·54 | — | — | — | 1 | — | — | — | — | — | — | 1 | | — | — | — | — | — | — | — | — | — | — | — | |
| | 16·5 | — | 3 | 6 | 2 | — | — | 11 | | — | 1 | — | — | — | — | — | — | — | — | 1 | | — | — | — | — | — | — | — | — | — | — | — | |

* Measured to nearest $\frac{1}{10}$ mm. † Calculated from tables with class intervals of ·25 cm. ‡ Stockport girls. § College students.

are reached a year or two earlier in both sexes. Some other relevant data are also available which have not been included in Table XIII; thus Dr F. M. Fonseca, Medical Officer of Health of Ebbw Vale, gives in his annual report for 1924 rates of prevalence at various ages suggesting a falling rate in girls after age 12, which if significant is exceptional.

Figure 5 shows that in girls there is a steady increase in goitre incidence till about the 16th year, little change from 16 to 19 and then a tendency to decrease after 19, as is indicated by Dr Turton's observations at Heanor, Derbyshire, which are the only continuous data of that kind I have found. In boys it seems certain that the maximum is reached earlier in most localities, though in Professor de Quervain's observations at Bern it was still apparently increasing at 16. My own observations in Somersetshire and Devonshire on some 2000 boys indicated a maximal incidence at age 12, and this is supported also by Dr Holroyd's observations on 3247 boys in Lancashire, by Dr A. V. Stocks' observations in Cheshire, by the findings of the School Medical Staff of the East Riding of Yorkshire, as well as by the Timaru figures referred to above. In the Isle of Wight however a significantly higher rate was found in boys at 13 than at 12.

All the observations used in constructing these curves are based on visual observation without measurement or palpation, that is by the same method essentially as was used by myself in classification by categories. Having obtained the mean breadth of the gland for each of my categories at various ages it is therefore possible, knowing the prevalence of enlargement according to my own standard at age 12 in a locality, to construct the mean curve of growth to be expected for thyroids in that locality. Thus if $g_0$ be the goitre rate according to my standard at age 12·5, $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$ the mean thyroid breadths for Categories 01, 2 and 3, at a given age $a$, then the mean breadths in that locality at age $a$ are given approximately by:

$$\text{Boys} \quad \bar{B} = (1 - zg_0)\,\bar{B}_1 + \cdot 98\ zg_0\bar{B}_2 + \cdot 02\ zg_0\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9\,a),$$

$$\text{Girls} \quad \bar{B} = (1 - zg_0)\,\bar{B}_1 + \cdot 818zg_0\bar{B}_2 + \cdot 182zg_0\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (9\,b),$$

where the respective values of $z$ are given by equations (8 $a$), (8 $b$) above. In Table XV are shown the thyroid breadth distributions at different ages, separating the three categories, and the mean breadths are shown in the same table. After smoothing by drawing freehand curves through these points the resulting values of $\bar{B}_1$, $\bar{B}_2$ and $\bar{B}_3$ are as shown in Table XVI.

To test the validity of the formulae (9) I have used the measurements made on the Stockport girls aged 9—17. These are set out in Table XVII, and for comparison a series of unselected Devonshire Boys* of ages 11—14 and Stockport boys aged 14—16 have been given in the same table. As shown at the end of Section (3) for this sample $g_0 = \cdot 262$, and substituting this value in equation (9 $b$) we obtain

$$\bar{B} = (1 - \cdot 262z)\,\bar{B}_1 + \cdot 214z\bar{B}_2 + \cdot 048z\bar{B}_3,$$

* The only locality where any selection was used in the Devonshire series was Chagford; hence the Chagford boys have been omitted from the complete table, which otherwise consists of the three category tables combined.

## TABLE XVI.

| Central Age | | 9·5 | 10·5 | 11·5 | 12·5 | 13·5 | 14·5 | 15·5 | 16·5 |
|---|---|---|---|---|---|---|---|---|---|
| Boys | $\bar{B}_1$ cm. | — | — | 3·42 | 3·42 | 3·42 | 3·43 | 3·54 | 3·54 |
| | $\bar{B}_2$ „ | — | — | 4·49 | 4·52 | 4·59 | 4·57 | 4·55 | 4·55 |
| | $\bar{B}_3$ „ | — | — | 7·27 | 7·27 | 7·27 | 7·27 | 7·27 | 7·27 |
| Girls | $\bar{B}_1$ „ | 3·29 | 3·29 | 3·29 | 3·30 | 3·30 | 3·60 | 4·01 | 4·03 |
| | $\bar{B}_2$ „ | 4·12 | 4·17 | 4·25 | 4·60 | 4·80 | 4·92 | 4·98 | 5·00 |
| | $\bar{B}_3$ „ | 7·12 | 7·12 | 7·12 | 7·10 | 7·06 | 6·90 | 6·90 | 6·90 |

## TABLE XVII.

*Breadth of Thyroid Gland in Random Sample Populations of Girls and Boys in Goitre Districts.*

| Central Age | Thyroid Breadth, cm. 2·25– | 2·75– | 3·25– | 3·75– | 4·25– | 4·75– | 5·25– | 5·75– | 6·25– | 6·75– | 7·25– | 7·75– | 8·25– | 8·75– | Totals | Mean Breadth* | No. with breadth over 42 mm. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *Stockport Girls* | | | | | | | | | | | | | | | | |
| 9·5 | 1 | 9 | 14 | 5 | 4 | — | — | — | — | — | — | — | — | — | 33 | 3·544 | 4 |
| 10·5 | — | 6 | 16 | 5 | 2 | — | — | 1 | — | — | — | — | — | — | 30 | 3·612 | 3 |
| 11·5 | — | 12 | 19 | 5 | 4 | — | 1 | — | — | — | 1 | — | 1 | — | 43 | 3·760 | 7 |
| 12·5 | — | 8 | 23 | 16 | 3 | — | — | — | 1 | — | 2 | — | — | 1 | 54 | 3·907 | 8 |
| 13·5 | — | 14 | 47 | 13 | 10 | 4 | 1 | 1 | 5 | 2 | — | 1 | 1 | — | 99 | 3·996 | 26 |
| 14·5 | — | 6 | 24 | 12 | 4 | 1 | 1 | 1 | 1 | — | — | — | 1 | — | 51 | 3·924 | 9 |
| 15·5 | — | 3 | 15 | 13 | 1 | — | 1 | 2 | 1 | 2 | — | 2 | — | 1 | 41 | 4·419 | 13 |
| 16·5 | — | — | 2 | 5 | 1 | — | 1 | 1 | — | — | — | 1 | — | — | 11 | 4·620 | 4 |
| 17·5 | — | — | 4 | — | 3 | 1 | — | — | — | 1 | — | 1 | — | — | 10 | 4·695 | 6 |
| | *Devonshire and Denbighshire Boys* | | | | | | | | | | | | | | | | |
| 10·75 | — | 7 | 4 | 1 | 2 | — | — | — | — | — | — | — | — | — | 14 | 3·461 | 2 |
| 11·25 | 4 | 9 | 18 | 18 | 11 | 5 | 2 | — | — | — | — | — | — | — | 67 | 3·844 | 39 |
| 11·75 | 5 | 12 | 18 | 18 | 7 | 9 | 1 | — | 1 | — | — | — | — | — | 71 | 3·835 | |
| 12·25 | 2 | 14 | 27 | 32 | 9 | 5 | 3 | 1 | 1 | — | — | — | — | — | 94 | 3·859 | 45 |
| 12·75 | 1 | 15 | 27 | 18 | 9 | 7 | 4 | — | — | — | — | — | — | — | 81 | 3·855 | |
| 13·25 | 3 | 14 | 28 | 23 | 14 | 7 | 1 | — | — | — | — | 1 | — | 1 | 92 | 3·916 | |
| 13·75 | 4 | 14 | 15 | 15 | 16 | 5 | 2 | 4 | 4 | 1 | 1 | — | — | — | 81 | 4·157 | 59 |
| 14·25 | 2 | 4 | 11 | 6 | — | 1 | — | — | — | — | — | 1 | — | — | 25 | 3·749 | 2 |
| 14·75 | — | — | 4 | 2 | 2 | — | — | — | — | — | — | — | — | — | 8 | | 2 |
| | *Stockport Boys* | | | | | | | | | | | | | | | | |
| 14·5 | — | 19 | 25 | 11 | 4 | — | 2 | — | — | 1 | — | — | — | — | 62 | 3·616 | — |
| 15·5 | — | 4 | 16 | 8 | 1 | 1 | — | 1 | — | — | — | — | — | — | 31 | 3·661 | — |
| 16·5 | — | 3 | 6 | 3 | — | — | — | — | — | — | — | — | — | — | 12 | | — |

* Calculated from tables with class intervals of ·25 cm.

and further substituting the appropriate values of $z$ from Table XIV at ages 9 to 16, this leads to the calculated breadths in Table XVIII.

TABLE XVIII.

| Age | *9·5* | *10·5* | *11·5* | *12·5* | *13·5* | *14·5* | *15·5* | *16·5* |
|---|---|---|---|---|---|---|---|---|
| Mean thyroid breadth calculated | 3·54 | 3·59 | 3·65 | 3·76 | 3·84 | 4·10 | 4·42 | 4·45 |
| Actual mean breadth found ... | 3·54 ± ·13 | 3·61 ± ·14 | 3·76 ± ·11 | 3·91 ± ·10 | 4·00 ± ·08 | 3·92 ± ·11 | 4·42 ± ·12 | 4·66 ± ·16 |

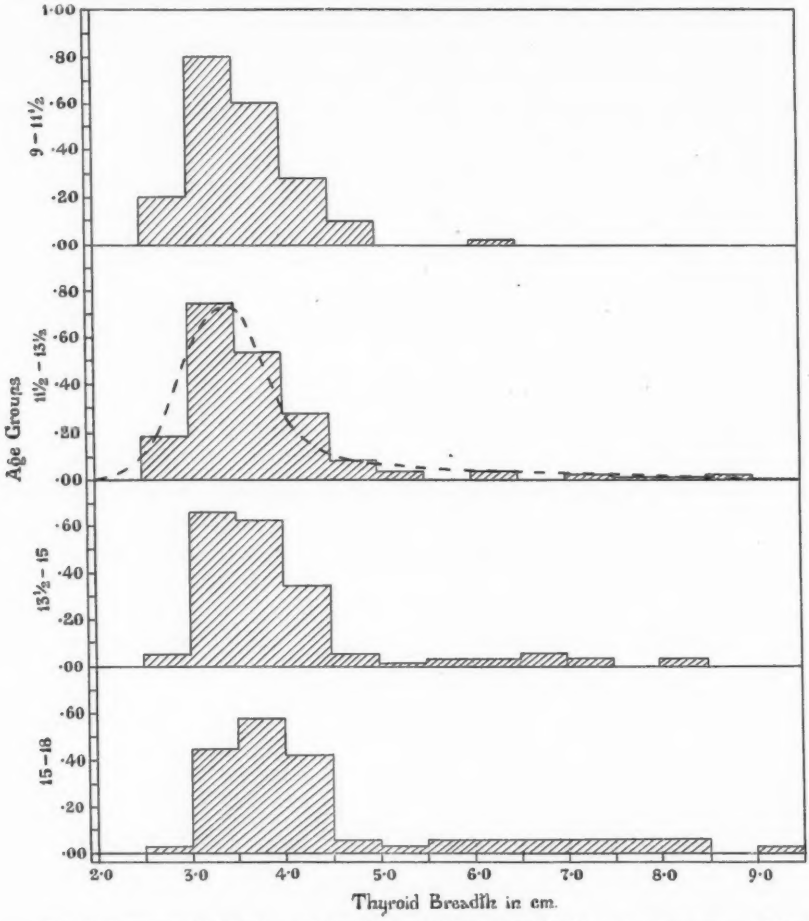



Fig. 6. Frequency distributions of thyroid breadth in Stockport girls at four age-groups. Distribution for ages 11½—13½ compared with frequency curve calculated from percentage at these ages having a thyroid breadth over 42 mm.

There is a reasonable measure of agreement between the calculated and actual means, which do not differ significantly at any age. The calculated curve is shown in Figure 7.

In the case of boys I have selected the town of Tiverton where I found in 154 boys, $\gamma_2 = \cdot225$, $\gamma_3 = \cdot011$, $g_0 = \cdot236$, at ages 11—13. Hence

$$\bar{B} = (1 - \cdot236z)\,\bar{B}_1 + \cdot225z\bar{B}_2 + \cdot011z\bar{B}_3,$$

and the resulting curve is shown in the lower part of Figure 7.

Hunziker (2) has tabulated the thyroid "superficial areas" of 353 untreated girls from a goitre region at ages 6—14, but these figures give us little information as to the *normal* growth of the thyroid, since both their actual mean values and the slope of the curve with age are functions of the local goitre prevalence and vary from one district to another. Thus the growth curves in Figure 7 for Stockport and Tiverton are only characteristic of those places or others with a similar goitre prevalence. The change in the form of frequency distribution with advancing age is illustrated in Figure 6, constructed from the Stockport data. The frequencies have been plotted for four age-groups, viz. 9—11½, 11½—13½, 13½—15, and 15—18, the actual distributions being as in Table XIX.

TABLE XIX.

*Central thyroid breadths in cm.*

| Age-group | 2·75 | 3·25 | 3·75 | 4·25 | 4·75 | 5·25 | 5·75 | 6·25 | 6·75 | 7·25 | 7·75 | 8·25 | 8·75 | 9·25 | Totals |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9—11½ | 8 | 32 | 24 | 11 | 4 | — | — | 1 | — | — | — | — | — | — | 80 |
| 11½—13½ | 12 | 49 | 35 | 18 | 5 | 3 | — | 3 | — | 2 | 1 | 1 | 2 | — | 131 |
| 13½—15 | 3 | 33 | 31 | 17 | 3 | 1 | 2 | 2 | 3 | 2 | — | 2 | — | — | 99 |
| 15—18 | 1 | 14 | 18 | 13 | 2 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | — | 1 | 62 |

In the diagram these have been reduced to a scale of unit total frequency in each case.

The next question is, *what constitutes normality* in the thyroid growth curves? No one who has examined schools in London and Stockport would contend for example that the Stockport mean curve should be taken as a criterion for normality. In contrast to this curve I have computed the curves to be expected in two other large towns, Bath and Liverpool.

At *Bath* I examined samples of 242 boys and 254 girls aged 11—13 and found for boys $\gamma_2 = \cdot080$, $\gamma_3 = \cdot003$, and for girls $\gamma_2 = \cdot208$, $\gamma_3 = \cdot011$, from which by equations (9), putting $g_0 = \gamma_2 + \gamma_3$, the approximate growth curves drawn in Figure 7 are obtained.

For *Liverpool* the following figures resulted from the combined observations of 14 medical officers:

Boys aged 12; examined 4914, thyroid enlarged 19, rate ·0039,
Girls aged 12; „ 4679, „ „ 70, „ ·0149.

It can be shown by calculation from the range of the individual personal equation that such rates based on 14 observers are liable to a considerable error. Regarding these rates as due to an average observer, however, and transferring to my own standard by means of equations (6), the rates would be $g_0 = \cdot 037$ for boys and ·067 for girls. The curve calculated on this basis for girls shows no appreciable increase in size of the gland from the 11th year to the 14th (see Figure 7) but a rapid increase from 14 to 16; the curve for boys does not differ sensibly from the "normal" curve and is not shown separately.

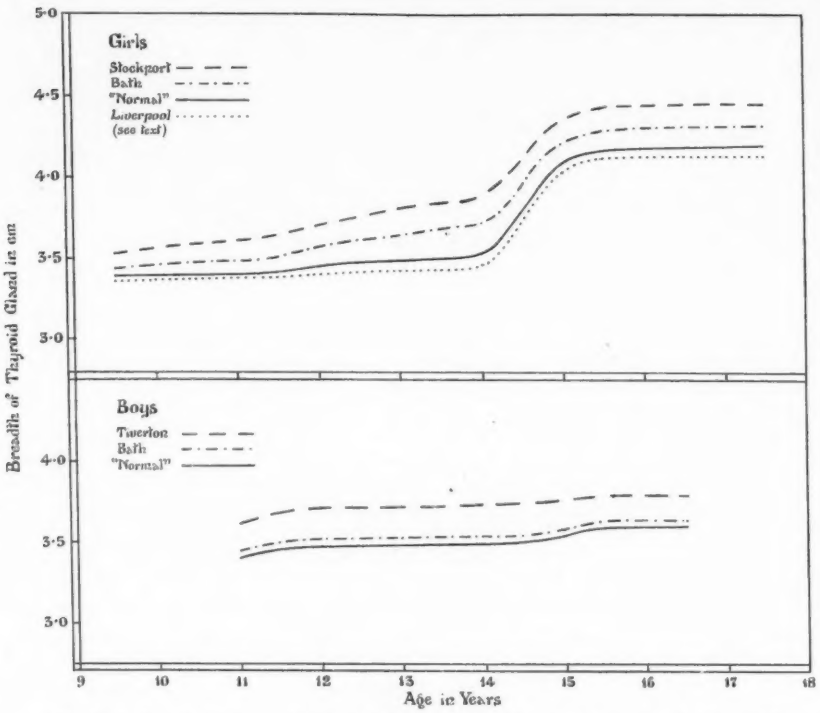



Fig. 7. Estimated mean growth curves of the thyroid gland in several towns, and "normal" growth curves for regions where goitre is not endemic.

Now Bath is situated in a region of rather high goitre prevalence and the Liverpool rates are subject to a considerable error. A better criterion on which to found a hypothetical "normal" growth curve is the mean rate for the whole of England and Wales outside the goitre area. After excluding all areas in which the goitre rate as determined by the 1924 survey was significantly high, we are left with a residual area in which 266,531 children were examined by 404 medical officers with resulting mean rates ·0149 for boys and ·0441 for girls. These

figures for the whole area, which have small probable errors, afford the best estimate of the amount of visible enlargement to be normally expected in a population of children aged about 12 and examined by an "average" observer. Transferring from the average standard to my own standard by equations (6) these rates are equivalent to $g_0 = \cdot0493$ for boys and $\cdot0977$ for girls, and on the same assumptions as before the *normal* curves of growth in thyroid breadth are approximately given by the equations

$$\text{Boys} \quad B = (1 - \cdot049z)\,\bar{B}_1 + \cdot048z\bar{B}_2 + \cdot001z\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots(10\,a),$$

$$\text{Girls} \quad \bar{B} = (1 - \cdot098z)\,\bar{B}_1 + \cdot080z\bar{B}_2 + \cdot018z\bar{B}_3 \quad \ldots\ldots\ldots\ldots\ldots(10\,b).$$

These curves have been drawn in Figure 7 and the values at half-yearly intervals are given below:

TABLE XX.

| Age | *9·5* | *10* | *10·5* | *11* | *11·5* | *12* | *12·5* | *13* | *13·5* | *14* | *14·5* | *15* | *15·5* | *16* | *16·5* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Boys | — | — | — | 3·40 | 3·47 | 3·47 | 3·48 | 3·48 | 3·48 | 3·48 | 3·48 | 3·53 | 3·59 | 3·59 | 3·59 cm. |
| Girls | 3·38 | 3·39 | 3·40 | 3·41 | 3·42 | 3·45 | 3·48 | 3·49 | 3·50 | 3·52 | 3·78 | 4·08 | 4·16 | 4·18 | 4·18 cm. |

The conclusion is that under normal conditions the thyroid gland scarcely changes at all in size in *boys* from $11\frac{1}{2}$ to 14, shows a slight increase from 14 to 16, and then remains stationary: from this it appears that the maximum size of the gland in boys has been almost attained by the 12th year, the relative increase in breadth from 11 to $15\frac{1}{2}$ being only 5·6 °/₀ as compared with 10 °/₀ for stature. In *girls* under normal conditions the thyroid undergoes very slight increase in size from 10 to $13\frac{1}{2}$ years but increases rapidly from the end of the 14th year to the middle of the 16th in the average girl, reaching the adult level about 16. To contrast its percentage rate of growth with the corresponding growth in stature during these periods, we have

| | *10—13½* | *13½—15½* | *10—15½* |
|---|---|---|---|
| Per cent. increase in thyroid breadth | 3·3 | 19 | 22·7 |
| Per cent. increase in stature ... | 14 | 7 | 21·5 |

The relative growth from 10 to $15\frac{1}{2}$ years is therefore about the same as for height, but whereas growth in height is roughly uniform during the period, that of the thyroid gland shows a pronounced retardation preceding puberty compensated for by a phenomenally rapid growth during puberty. As far as I am aware no other linear measurement hitherto taken on the body has revealed such remarkable prepubescent retardation.

(5) Relation to height and dimensions of head and neck.

(*a*) *Material.* In Cheshire the measurements of height, weight and grip, including the extraction of all previous records available, and the entire arrangements for the school visits, were carried out by Dr A. V. Stocks. In Devonshire, where annual height and weight records are kept of all children, the last of these records was used, the date being recorded; since all measurements had first to be corrected to age 12 from the age at which they were taken, any error involved by such a procedure would be trifling after age correction. The annual records extending back to age 4 or 5 were also extracted for use in Section (7).

The number of girls measured in Cheshire was 442, and this total was augmented to 540 by girls measured in London schools. Each series was corrected to age 12 from its own cubic age curve, and the resultant means and standard deviations did not differ significantly for any measurement except neck circumference to which reference has been made in Section (2), so there was no objection to combining them. The boys measured in Devonshire numbered 461, this total being augmented to 533 by boys from rural schools at Llanrwst, Llangerniew and Eglwysbach in Denbighshire.

The measurements taken were—in girls: height, weight, neck diameter and circumference, thyroid breadth and height of lobes, eye and hair colour, strength of grip, pulse rate, systolic and diastolic pressures and proficiency in class; in boys: height, weight, horizontal circumference of head, thyroid breadth and strength of grip. The boys measured were unselected but only random samples of the girls in the normal category were measured, as already explained in Section 3 (*a*). The totals measured in Category 01 and Categories 2 + 3 were finally in the proportions ·56 to 1. The actual proportion in which these categories occurred in the aggregate school population drawn upon was approximately 5·7 to 1, which implied that every 10th girl of category 01 had been measured. To ensure that a complete sample of the population was being employed for correlation, separate tables were made for these two category groups, the table for Category 01 being then multiplied by 10 and combined with the table for Categories 2 + 3. The probable errors of the correlation coefficients were calculated on the basis of the total girls measured and not on the augmented totals. It is to be noted that both for boys and girls the group consisting of Categories 2 + 3 is more homogeneous than the completed sample in the sense that all the children in it had been exposed to the external goitre-producing factor whereas in the complete population this may not have been so and all may not have had the same chance of developing thyroid enlargement. In any case the complete sample is heterogeneous in the sense that it consists of a mixture of "normals" with "abnormals," and since the proportions in which these are mixed would differ in samples from different districts, the correlation coefficients for the complete samples would also vary, and those here calculated are only applicable to the districts sampled in this particular instance. For this and other reasons correlation coefficients have been worked out separately in all the succeeding sections (i) for Category

01 group, (ii) for Categories 2 + 3, (iii) for the complete sample, the last of these being of only limited value owing to its dependence on goitre prevalence.

(*b*) *Correction to age* 12. The correction of the thyroid breadth for age was first carried out by means of the age curves for the Category 01 group, the assumption being that since this represented its growth when not visibly enlarged, it was the best base line from which to compute degrees of enlargement. The points on this curve are tabulated along the lines $\bar{B}_1$ in Table XVI in Section (4). It was later realised that a better curve from which to calculate age corrections would be the "normal" growth curve deduced in Figure 7. For boys the resulting age corrections were so nearly identical that it made no difference which curve was used, but for girls the curves were appreciably different and hence all the series of coefficients with thyroid breadth corrected from the "normal" growth curve were calculated and are shown in Table XXVI; the coefficients do not differ significantly from the others. For brevity the Category 01 growth curve will be referred to as *B*1 and the "normal" growth curve as *BN*. The two series of age corrections at half-year periods were as follows, a rule representing zero and a dot no data:

TABLE XXI.

*Corrections to age* 12 *exactly, in cm.*

| | | *10* | *10·5* | *11* | *11·5* | *12* | *12·5* | *13* | *13·5* | *14* | *14·5* | *15* | *15·5* | *16* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Boys | *B*1 | · | +·32 | +·09 | — | — | — | — | — | — | — | — | · | · |
| | *BN* | · | +·32 | +·08 | +·01 | — | — | — | — | — | — | −·02 | · | · |
| Girls | *B*1 | +·02 | +·02 | +·01 | +·01 | — | — | — | — | −·01 | −·27 | −·62 | −·73 | −·75 |
| | *BN* | +·06 | +·05 | +·04 | +·03 | — | −·03 | −·04 | −·05 | −·07 | −·33 | −·63 | −·71 | −·73 |

(*c*) *Correlation with Height.* The growth curves for height and weight were constructed from the complete records of the children measured, going back in some cases to three years of age. For boys the total entries of height were 2438, of weight 2494; in girls the total entries were 1401 and 1395 respectively. The complete series of mean heights and weights are shown in Table XXII and to these cubic curves were fitted with the formulae

$$\text{Boys} \quad H = 72{\cdot}635 + 7{\cdot}3096a - {\cdot}23739a^2 + {\cdot}008092a^3 \ldots\ldots\ldots\ldots\ldots (11\,a),$$

$$\text{Girls} \quad H = 70{\cdot}857 + 6{\cdot}6641a - {\cdot}07158a^2 + {\cdot}000633a^3 \ldots\ldots\ldots\ldots\ldots (11\,b),$$

where $H$ = height in cm., $a$ = age in years.

These curves are drawn in Figure 8, the scale unit being the height at age 11 exactly. The actual mean heights reduced from Table XXII to this scale are shown also in Figure 8 as black dots in order to demonstrate that the cubic curves closely fit the data and have not had the effect of smoothing out pubescent irregularities to any appreciable extent. The cubic curves have been carried only to age 15, after which the curves have been continued in a freehand manner to

age 20 by means of other data in the laboratory. On the same diagram are shown the thyroid breadth curves, the scale unit being the breadth at age 11;

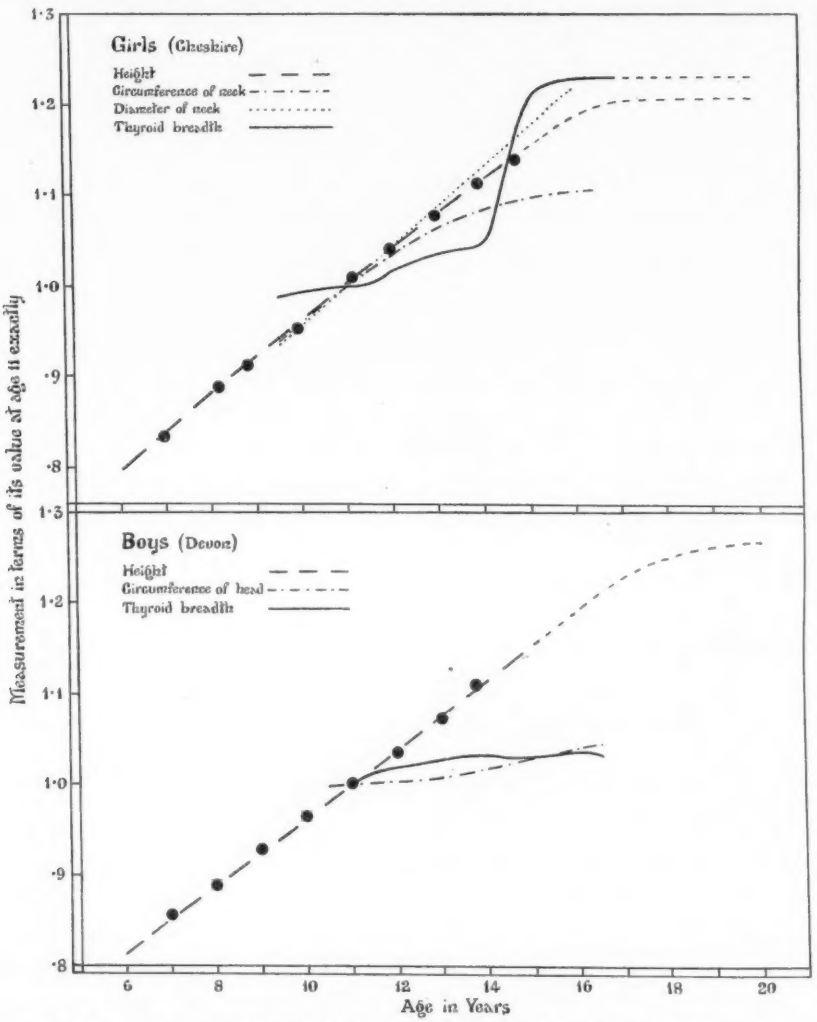



Fig. 8. Growth curves of thyroid gland contrasted with growth curves for other linear measurements in the same districts, the mean measurements at age 11 exactly being taken as scale unit in each case.

these curves are arrived at in the same way as the curves for Bath in Figure 7 and Section (4), and represent the average growth curves for the thyroid in the

school populations which were sampled for the purpose of this and the following sections. Their formulae, before converting to the scale unit of breadth at age 11, were

$$\text{Boys} \quad \bar{B} = (1 - \cdot 386z)\,\bar{B}_1 + \cdot 378z\bar{B}_2 + \cdot 008z\bar{B}_3,$$
$$\text{Girls} \quad \bar{B} = (1 - \cdot 150z)\,\bar{B}_1 + \cdot 123z\bar{B}_2 + \cdot 027z\bar{B}_3,$$

where the values of $\bar{B}_1$, $\bar{B}_2$, $\bar{B}_3$, $z$ are tabulated in Tables XVI and XIV. The contrast with growth in height is readily seen when the curves are reduced to the same relative scale as in Figure 8.

Correction of height for age was carried out from curves (11) above; points on these curves from 10 to 16 are given in Table XXIII. After age correction the correlation coefficients between thyroid breadth and stature were

$$\text{Boys} \quad r = \cdot 1371 \pm \cdot 0294. \quad \text{Girls} \quad \begin{cases} B1 \text{ curve } r = \cdot 2141 \pm \cdot 0298, \\ BN \quad \text{,,} \quad r = \cdot 2222 \pm \cdot 0291. \end{cases}$$

The values of $\eta$ for stature on breadth were for boys $\eta = \cdot 2227$, girls $\eta = \cdot 2787$, these values being significant as are also the correlation coefficients above; thus for boys $\eta^2 - \overline{\eta}^2 = \cdot 02392$ and the probable error of $\overline{\eta}^2 = \cdot 0067$; the corresponding values for girls being ·07151 and ·0034. The mean heights at intervals of thyroid breadth are shown in Table XXIV.

Cubic curves have been fitted as regression lines of height on thyroid size, having the equations

$$\text{Boys} \quad H' = 91 \cdot 854 + 27 \cdot 7527B' - 4 \cdot 98536B'^2 + \cdot 281859B'^3,$$
$$\text{Girls} \quad H' = 106 \cdot 445 + 19 \cdot 4829B' - 3 \cdot 35291B'^2 + \cdot 191280B'^3,$$

where $H'$ and $B'$ are statures and thyroid breadths in cm. corrected to age 12. These are drawn in Figure 9 and it is evident that the slope of the regression lines is pronounced whilst the thyroid is not sensibly enlarged, but with moderate enlargement it becomes horizontal; for girls it again turns upwards when enlargement becomes considerable, but this last relation is doubtful in boys, the last point in the figure representing only one case. The correlations within the separate category groups—including respectively 310 and 196 boys and 180 and 306 girls—are as follows:

| | | |
|---|---|---|
| Category 01. | Boys $r = \cdot 2045 \pm \cdot 0367$, | Girls $(B1)\, r = \cdot 2909 \pm \cdot 0485$, |
| Categories 2 + 3. | ,, $r = \cdot 0031 \pm \cdot 0482$, | ,, $r = \cdot 2251 \pm \cdot 0366$. |

If the size of the thyroid has no relation to height other than that arising from simultaneous growth in all linear measurements we should expect to find a positive correlation amongst the Category 01 or normal groups, and that this would become almost obliterated in passing to the group of goitrous children since the small variations in thyroid size due to height would become lost in the much larger variations due to the goitre factor. This is what actually occurs in boys, the coefficient being reduced from ·2045 to ·0031, but in girls it is initially larger, viz. ·2909, and is only reduced to ·2251 in the goitre group. This suggests that

whereas in boys the correlation is entirely due to the natural relationship existing between any two linear measurements in groups of children of the same age, in girls there is an additional relation involved which becomes more pronounced as the thyroid becomes larger.

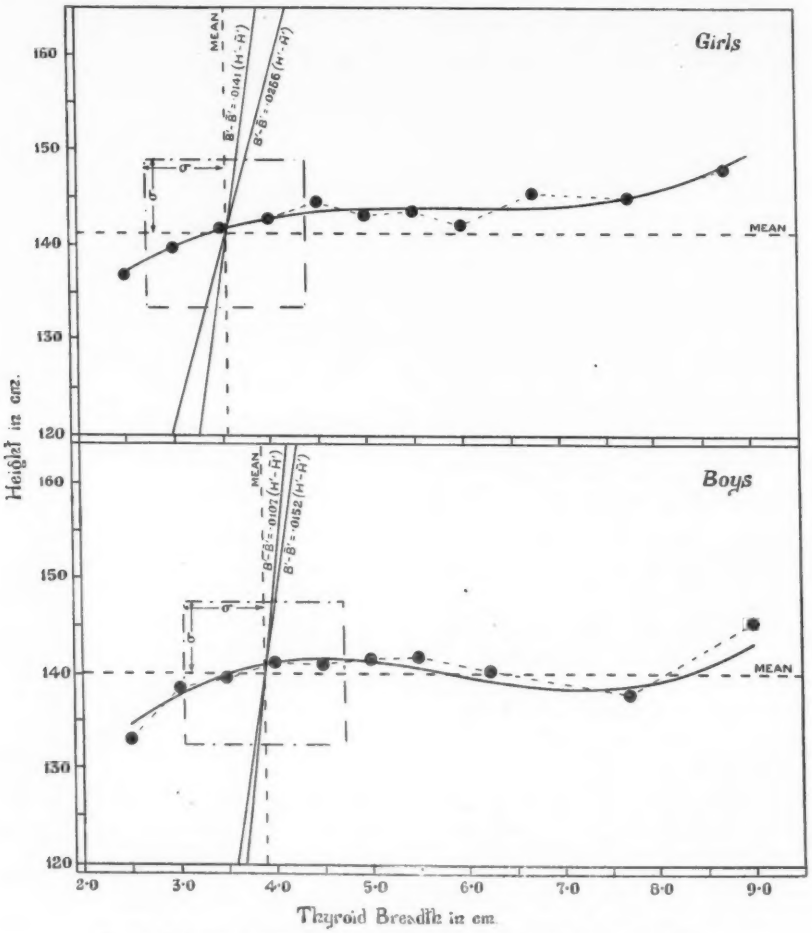



Fig. 9. Relations between thyroid breadth and height, both corrected to age 12.

It is this additional relation that we are concerned in discovering in this research; that is, *whether enlargement of the thyroid gland beyond the dimensions expected for the age and height of the child measured tends to be associated with a height or other physical measurement above or below average.*

What breadth of thyroid is to be expected at age 12 for a given stature? This can only be ascertained by obtaining a measure of the relation between thyroid breadth and stature in children whose thyroid glands are functioning normally. It is obvious that the regression line of thyroid breadth on height for

TABLE XXII.

*Records of Height and Weight in Devonshire Boys and Cheshire Girls whose Thyroid Glands were measured.*

| Central Age | Boys | | | | Girls | | | |
|---|---|---|---|---|---|---|---|---|
| | Height | | Weight | | Height | | Weight | |
| | Total entries | Mean height, cm. | Total entries | Mean weight, kgm. | Total entries | Mean height, cm. | Total entries | Mean weight, kgm. |
| *3·25* | — | — | — | — | 14 | 91·02 | 15 | 14·05 |
| *3·37* | 45 | 95·45 | 46 | 15·91 | — | — | — | — |
| *3·75* | — | — | — | — | 9 | 93·78 | 9 | 15·78 |
| *4·25* | — | — | — | — | 25 | 98·29 | 25 | 16·17 |
| *4·55* | 96 | 101·20 | 98 | 17·16 | — | — | — | — |
| *4·75* | — | — | — | — | 29 | 101·38 | 30 | 17·25 |
| *5·25* | 77 | 105·90 | 87 | 18·42 | 60 | 104·45 | 59 | 17·18 |
| *5·75* | 86 | 108·12 | 100 | 18·69 | 67 | 106·05 | 68 | 18·23 |
| *6·25* | 95 | 111·00 | 96 | 19·98 | 53 | 111·15 | 53 | 19·41 |
| *6·75* | 95 | 113·93 | 96 | 20·76 | 38 | 112·61 | 37 | 19·58 |
| *7·25* | 111 | 116·45 | 115 | 21·89 | 23 | 115·06 | 22 | 20·36 |
| *7·75* | 126 | 118·31 | 130 | 22·88 | 20 | 121·15 | 21 | 23·45 |
| *8·25* | 160 | 121·23 | 158 | 23·79 | 103 | 120·93 | 103 | 23·08 |
| *8·75* | 156 | 124·08 | 158 | 24·96 | 137 | 124·19 | 129 | 23·87 |
| *9·25* | 160 | 126·06 | 159 | 25·92 | 37 | 124·93 | 38 | 25·00 |
| *9·75* | 150 | 129·41 | 156 | 27·41 | 15 | 128·65 | 15 | 25·45 |
| *10·25* | 150 | 131·23 | 156 | 28·45 | 10 | 131·65 | 9 | 27·78 |
| *10·75* | 174 | 134·19 | 179 | 30·00 | 12 | 136·70 | 11 | 31·10 |
| *11·25* | 169 | 136·56 | 169 | 31·20 | 65 | 137·79 | 65 | 30·17 |
| *11·75* | 165 | 138·25 | 165 | 32·94 | 110 | 139·91 | 112 | 32·16 |
| *12·25* | 161 | 141·57 | 161 | 34·57 | 165 | 143·05 | 166 | 34·74 |
| *12·75* | 115 | 142·61 | 119 | 34·96 | 138 | 145·86 | 137 | 37·63 |
| *13·25* | 89 | 148·01 | 88 | 38·53 | 110 | 147·90 | 110 | 38·88 |
| *13·75* | 58 | 150·26 | 43 | 40·36 | 92 | 151·12 | 92 | 40·85 |
| *14·25* | — | — | 11 | 41·36 | 54 | 153·11 | 54 | 43·06 |
| *14·75* | — | — | 4 | 45·45 | 9 | 155·12 | 9 | 43·79 |
| *15·25* | — | — | — | — | 4 | 152·95 | 4 | 46·95 |
| *15·75* | — | — | — | — | 2 | 152·95 | 2 | 43·95 |
| Totals | 2438 | — | 2494 | — | 1401 | — | 1395 | — |

the whole sample, including the goitrous, will not give this information, and correction of breadth from this line would be merely begging the question. The nearest approximation to the relation we require is given by the regression line of thyroid breadth on stature, both corrected to age 12, in the Category 01 group of children who presented no obvious enlargement. In using this line to correct from, the

TABLE XXIII. *Mean Physical Measurements and Smoothed Values at Half-year Periods in Devonshire Boys and Cheshire Girls.*

| Central Age | Boys | | | | | | | | Girls | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Height, cm. from curve | Weight, kgm. from curve | Head circumference, mm. | | | Strength of grip, kgm. | | | Height, cm. from curve | Weight, kgm. from curve | Strength of grip, kgm. | | | Blood pressure : mm. of mercury | | | | | | Pulse interval, seconds | | |
| | | | | | | | | | | | | | | Systolic | | | Diastolic | | | Pulse | | | |
| | | | No. | Mean | Smoothed | No. | Mean | Smoothed | | | No. | Mean | Smoothed | No. | Mean | Smoothed | No. | Mean | Smoothed | Smoothed | No. | Mean | Smoothed |
| *10·25* | — | — | — | — | — | — | — | — | 132·32 | 28·05 | 1 | } 14·25 | 13·3 | 1 | } 107·0 | 101·5 | 1 | } 69·1 | 66·6 | 34·9 | 1 | } ·677 | ·667 |
| *10·75* | 133·83 | 29·87 | 14 | 525·6 | 525 | 14 | 14·79 | 14·7 | 135·01 | 29·57 | 4 | | 14·1 | 4 | | 104·4 | 3 | | 68·0 | 36·4 | 4 | | ·672 |
| *11·25* | 136·34 | 31·31 | 68 | 528·6 | 526 | 69 | 16·44 | 16·4 | 137·67 | 31·19 | 36 | | 14·9 | 36 | | 107·6 | 35 | | 69·3 | 38·3 | 36 | | ·676 |
| *11·75* | 138·87 | 32·83 | 72 | 526·1 | 527 | 72 | 17·92 | 17·9 | 140·30 | 32·91 | 43 | 16·28 | 15·7 | 43 | } 112·1 | 110·5 | 34 | } 71·3 | 70·6 | 39·9 | 43 | | ·678 |
| *12·25* | 141·43 | 34·43 | 95 | 530·5 | 529 | 95 | 19·11 | 19·1 | 142·91 | 34·73 | 48 | 16·48 | 16·5 | 49 | | 113·4 | 43 | | 72·0 | 41·4 | 49 | ·680 | ·679 |
| *12·75* | 144·01 | 36·12 | 83 | 527·5 | 530 | 83 | 20·96 | 20·1 | 145·50 | 36·65 | 61 | 17·34 | 17·3 | 62 | 115·9 | 115·8 | 53 | } 73·4 | 73·0 | 42·8 | 62 | ·675 | ·685 |
| *13·25* | 146·63 | 37·91 | 92 | 533·1 | 532 | 92 | 20·46 | 21·3 | 148·06 | 38·68 | 92 | 18·02 | 18·0 | 98 | 116·6 | 116·7 | 85 | | 73·5 | 43·2 | 99 | ·697 | ·691 |
| *13·75* | 149·30 | 39·79 | 82 | 535·5 | } 535 | 82 | 23·84 | 23·1 | 150·60 | 40·83 | 88 | 18·89 | 18·9 | 131 | 117·4 | 117·4 | 114 | 73·3 | 73·3 | 44·1 | 131 | ·696 | ·689 |
| *14·25* | 152·01 | 41·79 | 26 | 533·5 | | 26 | 24·74 | 25·0 | 153·12 | 43·09 | 54 | 21·06 | 21·0 | 73 | 119·7 | 119·7 | 70 | 73·1 | 73·1 | 46·6 | 73 | ·649 | ·660 |
| *14·75* | 154·77 | 43·87 | 8 | | | 8 | | 27·0 | 155·61 | 45·47 | 10 | } 22·87 | 22·3 | 20 | } 120·3 | 120·5 | 19 | 73·2 | 73·2 | 47·3 | 20 | ·655 | ·652 |
| *15·25* | — | — | — | — | — | — | — | — | 158·08 | 47·97 | 4 | | 23·4 | 10 | | 121·0 | 10 | } 76·7 | 74·5 | 46·5 | 9 | } ·669 | ·659 |
| *15·75* | — | — | — | — | — | — | — | — | 160·53 | 50·60 | 2 | | 24·4 | 12 | 122·2 | 121·3 | 11 | | 78·0 | 43·3 | 12 | | ·679 |

## TABLE XXIV.

*Relations of Physical Measurements to Breadth of the Thyroid Gland, both corrected to age* 12.—*Means and Frequencies**.

| Central Thyroid Breadth, cm. | Boys | | | | Girls | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Height, cm. | | Weight, kgm. | | Height, cm. | | Weight, kgm. | | Rate of growth†, cm. per year | | Rate of increase† in weight, kgm. per year | | Strength of grip, kgm. | | Systolic pressure, mm. | | Diastolic pressure, mm. | | Pulse pressure, mm. | | Pulse interval, seconds | | Proficiency in class | |
| | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean | No. | Mean |
| 2·5 | 17 | 133·10 | 19 | 28·87 | 25 | 136·93 | 24 | 29·95 | 23 | − ·217 | 22 | − ·575 | 28 | 15·64 | 28 | 108·4 | 27 | 63·4 | 27 | 41·8 | 28 | ·672 | 27 | − ·191 |
| 3·0 | 82 | 138·64 | 78 | 33·06 | 58 | 139·75 | 57 | 32·55 | 56 | + ·073 | 56 | − ·162 | 66 | 16·66 | 66 | 109·0 | 55 | 70·4 | 55 | 39·5 | 65 | ·657 | 59 | − ·141 |
| 3·5 | 147 | 139·71 | 143 | 33·22 | 99 | 141·95 | 97 | 33·60 | 92 | − ·014 | 92 | + ·072 | 109 | 15·97 | 109 | 111·1 | 96 | 71·1 | 96 | 41·3 | 109 | ·673 | 106 | + ·040 |
| 4·0 | 126 | 141·14 | 127 | 34·37 | 84 | 142·80 | 82 | 33·97 | 71 | + ·298 | 68 | + ·607 | 89 | 16·12 | 89 | 111·4 | 76 | 71·1 | 76 | 41·7 | 88 | ·678 | 83 | + ·177 |
| 4·5 | 69 | 140·97 | 67 | 33·85 | 68 | 144·56 | 68 | 36·82 | 65 | − ·543 | 66 | − ·020 | 71 | 19·66 | 70 | 108·4 | 64 | 72·3 | 64 | 39·6 | 70 | ·691 | 66 | + ·074 |
| 5·0 | 40 | 141·55 | 40 | 33·60 | 39 | 143·07 | 39 | 35·30 | 37 | − ·002 | 38 | − ·183 | 42 | 17·50 | 43 | 112·6 | 37 | 71·6 | 37 | 42·8 | 43 | ·687 | 38 | − ·059 |
| 5·5 | 10 | 141·85 | 10 | 35·25 | 22 | 143·68 | 22 | 35·50 | 20 | − ·234 | 20 | + ·407 | 24 | 17·75 | 24 | 112·8 | 23 | 69·0 | 23 | 43·9 | 24 | | 22 | + ·536 |
| 6·0 | 5 | 141·85 | 5 | 33·45 | 30 | 142·25 | 30 | 34·15 | 28 | + ·066 | 29 | + ·142 | 30 | 16·57 | 30 | 116·7 | 26 | 73·7 | 26 | 45·5 | 30 | ·689 | 28 | + ·059 |
| 6·5 | 5 | 138·85 | 5 | 34·05 | 13 | 144·07 | 13 | 35·52 | 11 | + ·020 | 12 | + ·526 | 13 | 17·88 | 13 | 113·1 | 13 | 73·9 | 13 | 38·9 | 13 | ·647 | 13 | − ·499 |
| 7·0 | 1 | 139·45 | 1 | 33·45 | 14 | 146·74 | 14 | 37·30 | 12 | + ·233 | 12 | + ·357 | 14 | 18·21 | 13 | 111·6 | 11 | 72·4 | 11 | 41·5 | 14 | ·664 | 14 | +1·166 |
| 7·5 | 1 | 139·45 | 1 | 30·45 | 17 | 146·16 | 17 | 38·21 | 17 | + ·213 | 17 | +1·078 | 17 | 19·21 | 17 | 117·9 | 17 | 74·9 | 17 | 43·2 | 17 | ·659 | 17 | + ·221 |
| 8·0 | 2 | 136·45 | 2 | 33·45 | 8 | 142·45 | 8 | 36·45 | 7 | + ·209 | 7 | +1·178 | 8 | 18·50 | 8 | 117·6 | 7 | 78·4 | 7 | 42·7 | 8 | | 8 | + ·809 |
| 8·5 | — | — | — | — | 7 | 148·88 | 7 | 39·45 | 7 | | 6 | +1·278 | 7 | 19·17 | 7 | 113·4 | 6 | 73·7 | 6 | 42·0 | 7 | ·688 | 6 | +1·059 |
| 9·0 | 1 | 145·45 | 1 | 42·45 | 2 | 145·45 | 2 | 37·95 | 2 | + ·566 | 2 | | 2 | | 2 | 117·0 | 2 | 74·5 | 2 | 39·5 | 2 | | 2 | |

* In the case of the girls these are the actual numbers measured; for correlation the frequencies have been adjusted as explained in the text.

† In excess or defect of the average rates given by the cubic curves in the same interval.

assumption is that no *causal* association between thyroid size and stature will begin to come into operation until the gland was noticeably enlarged; such an assumption seems reasonable, and at any rate if any causal association is also present in the normal group it means that we are over-correcting for height and reducing all the positive coefficients too far, so that any conclusions drawn as to the significance of positive coefficients between thyroid breadth and physical characters will not be invalidated.

The two regression lines on stature, both for boys and girls, are drawn in Figure 9 for comparison; they scarcely show any significant departure from linearity for the complete samples, and still less for Category 01 alone. For the complete samples of boys $\eta = \cdot2093$, $\eta^2 - \overline{\eta}^2 = \cdot01220$, p.e. of $\overline{\eta}^2 = \cdot00741$, and of girls $\eta = \cdot3126$, $\eta^2 - \overline{\eta}^2 = \cdot09057$, p.e. of $\overline{\eta}^2 = \cdot00363$. Thus $\eta^2$ is non-significant for the boys but is significant for the girls. The data for obtaining the regression coefficients are

TABLE XXV.

| | | Mean Breadth $B'$ | $\sigma_{B'}$ | Mean Stature $H'$ | $\sigma_{H'}$ |
|---|---|---|---|---|---|
| Boys | Complete sample ... | 3·8972 cm. | ·8327 cm. | 140·043 cm. | 7·495 cm. |
| | Category 01 only ... | 3·4192 „ | ·3915 „ | 139·266 „ | 7·501 „ |
| Girls | Complete sample ... | 3·6922 „ | ·9129 „ | 141·487 „ | 7·550 „ |
| | Category 01 only ... | 3·4086 „ | ·3761 „ | 141·246 „ | 7·765 „ |

and the equations of the four regression lines become

Boys { Complete sample. $(B' - 3{\cdot}8972) = {\cdot}01523\,(H' - 140{\cdot}043)$ cm. per cm.
Category 01 only. $(B' - 3{\cdot}4192) = {\cdot}01067\,(H' - 139{\cdot}266)$ „

Girls { Complete sample. $(B' - 3{\cdot}6922) = {\cdot}02589\,(H' - 141{\cdot}487)$ „
Category 01 only. $(B' - 3{\cdot}4086) = {\cdot}01409\,(H' - 141{\cdot}246)$ „

Correction of thyroid breadths from the regression lines for the complete samples has of course the effect of reducing the correlation with height in the complete samples to zero. Correction from the regression lines for normal children only likewise reduces the correlation with height in the Category 01 group to zero, and what we want to determine is whether, after such correction of every individual thyroid breadth, we are left with any significant correlation in the goitrous and complete samples. The standard deviations of thyroid breadth after correction from the regression lines for Category 01, which may be termed $\sigma_{B''}$, and the standard deviations of height, $\sigma_{H'}$, in these two samples are

Boys { Complete sample. $\sigma_{B''} = {\cdot}8255$ cm., $\sigma_{H'} = 7{\cdot}4947$ cm.
Categories 2 + 3. $\sigma_{B''} = {\cdot}7877$ „ $\sigma_{H'} = 7{\cdot}3185$ „

Girls { Complete sample. $\sigma_{B''} = {\cdot}8973$ „ $\sigma_{H'} = 7{\cdot}5496$ „
Categories 2 + 3. $\sigma_{B''} = 1{\cdot}3250$ „ $\sigma_{H'} = 6{\cdot}7765$ „

The correlation coefficient of thyroid breadth, so corrected, with height, is then given by $r_{B''H'} = \frac{1}{\sigma_{B''}}(\sigma_{B'}r_{B'H} - b\sigma_{H'})$*, where $b$ is the regression coefficient found above for $B'$ on $H'$ in the Category 01 group, that is, where $b = \cdot01067$ for boys and $\cdot01409$ for girls. The resulting values of $r_{B''H'}$ are

| | | |
|---|---|---|
| Boys | Complete sample. | $r = \cdot0414 \pm \cdot0299$. |
| | Categories 2 + 3. | $r = -\cdot0960 \pm \cdot0477$. |
| Girls | Complete sample. | $r = \cdot0993 \pm \cdot0309$. |
| | Categories 2 + 3. | $r = \cdot1561 \pm \cdot0376$. |

In the *boys* there remains no significant correlation over and above that which is attributable to the normal degree of proportionality between linear measurements at a constant age. If anything the relation becomes negative in the goitrous group, which might be due to over correction by the assumption made above, but it is not significantly negative.

In the *girls* there remains a significant positive correlation over and above that due to proportionality and this becomes more pronounced when attention is confined to the goitrous group alone as might be expected from the regression line on breadth. This indicates that either (i) goitres tend to appear more readily or to become more pronounced relative to the child's size in girls whose height is above average for their age, or else (ii) girls whose thyroids enlarge about puberty tend in consequence to grow more rapidly in height. If the first be the correct interpretation, the natural explanation is that a girl who is taller than average, or is growing more rapidly than average, requires more thyroid secretion than the average girl and more than her size of gland is capable of supplying unless the manufacture of the essential factor in the secretion from external sources can keep up with the demand. Usually the intake of iodine for this purpose would be abundant enough to meet extra demands, but in goitrous regions where evidence suggests that it is barely sufficient, the girls who require most would be the first to exhibit signs of deficiency by compensatory enlargement of the gland.

If the second interpretation be the correct one it would mean that enlargement of the gland about puberty is caused by some infective, nervous or other stimulus which itself also stimulates growth or else that the altered thyroid activity accompanying enlargement is responsible for more rapid growth; such an explanation is highly improbable and is in my view quite disproved by evidence in the succeeding sections. This important question will be further discussed at the end of Section (7).

(*d*) *Partial correlation to eliminate the effect of height.* The calculations used above may be further applied to get rid of the height factor from the correlations

* Since the values of $B''$ are corrected from the regression line, breadth $= bH'$, for an array at height $H'$ we have $B'' = B' - bH'$; hence from the product-moment formula

$$r_{B''H'}\sigma_{B''}\sigma_{H'} = \frac{1}{N}\Sigma[B'' \times H'] = \frac{1}{N}\Sigma[(B' - bH')\,H'] = r_{B'H'}\sigma_{B'}\sigma_{H'} - b\sigma^2_{H'}.$$

between thyroid breadth and other physical measurements*. In dealing with the Category 01 group alone the ordinary method of partial correlation for height constant can be used since by so doing we are correcting from the regression lines of both factors on height within the normal group. In dealing with the Categories 2 + 3 group or the complete sample, however, it is desired to correct for only that part of the correlation arising from normal proportionality, that is, to correct thyroid breadth from the regression line on height for the normal group. The regression and correlation coefficients of other physical factors with height do not differ significantly in the three groups as can be seen by reference to Table XXVI. Hence the formula for partial correlation with some factor $X'$ (corrected to age 12) becomes

$$\frac{r_{B'X'} - r_{X'H'} \cdot b\frac{\sigma_{H'}}{\sigma_{B'}}}{\sqrt{1 - r^2_{B'X'}}\sqrt{1 - b^2\frac{\sigma^2_{H'}}{\sigma^2_{B'}}}},$$

where $b$ is the regression coefficient of $B'$ on $H'$ for the Category 01 group and has in all cases the values ·01067 for boys and ·01409 for girls, whilst $\sigma_{H'}$, $\sigma_{B'}$ and $r_{B'X'}$ are the values for the group being dealt with. For Category 01 group alone $b\frac{\sigma_{H'}}{\sigma_{B'}} = r_{B'H'}$ for that group, but for the Categories 2 + 3 group or the complete sample it has somewhat different values as follows:

$b\frac{\sigma_{H'}}{\sigma_{B'}}$ { Categories 2 + 3. Boys ·0996, Girls ·0711.
Complete sample. „ ·0960, „ ·3165.

These coefficients will therefore be used for partial correlation in the following sections to eliminate the normal effect of height.

(*e*) *Correlation with dimensions of neck.* The neck measurements have been described in Section 2 (*c*). Their growth curves are shown in Figure 8 where the measurements are reduced to a proportionate scale; the contrast with the thyroid curve is evident. The correlation coefficients after age correction are shown in Table XXVI. In order to correct for height the regressions on height are required in the normal groups alone, since neck measurements are obviously affected by large goitres, and these are:

| | | | |
|---|---|---|---|
| Antero-posterior diameter. | ·2425 mm. | per cm. | of height. |
| Neck circumference. | ·8052 „ | „ | „ |

Passing to the goitre groups and complete samples, we have:

Categories 2 + 3. $\sigma_{D'} = 7{\cdot}5251$ mm., $\sigma_{C'} = 20{\cdot}9867$ mm., $\sigma_{H'} = 6{\cdot}7765$ cm.,
Complete sample. $\sigma_{D'} = 5{\cdot}8697$ mm., $\sigma_{C'} = 19{\cdot}2662$ mm., $\sigma_{H'} = 7{\cdot}5496$ cm.,

* It is not suggested that by making age and height constant the effect of size is completely eliminated; it must be admitted that height would not perhaps be naturally so highly correlated with thyroid breadth as would some horizontal measurements on the body such as breadth of neck or shoulders, but as it was not practicable to include such measurements correction for height has been used here, the effect of this being to limit the field to children of equal age and height and to that extent exclude natural correlations due to size.

## TABLE XXVI.

*Correlation coefficients between Breadth of Thyroid Gland and various Physical and Mental Characters, all corrected for age.*

| Boys | With Thyroid Breadth corrected for age* | | | | | | With height corrected for age | | |
|---|---|---|---|---|---|---|---|---|---|
| Factors corrected for age | Without correction for height | | | After elimination of normal correlation due to height† | | | | | |
| | Category 01 | Categories 2+3 | All boys | Category 01 | Categories 2+3 | All Boys | Category 01 | Categories 2+3 | All boys |
| Height ... ... ... | ·2045 ± ·0367 | ·0031 ± ·0482 | ·1528 ± ·0293 | — | − ·0960 | ·0414 | — | — | — |
| Weight ... ... ... | ·3048 ± ·0352 | ·0396 ± ·0482 | ·1187 ± ·0298 | ·2394 | − ·0863 | ·0701 | ·7938 ± ·0143 | ·8509 ± ·0133 | ·8069 ± ·0105 |
| Rate of growth ... ... | ·0527 ± ·0470 | − ·0407 ± ·0543 | ·0176 ± ·0356 | − ·0336 | − ·0852 | − ·0212 | ·4049 ± ·0394 | ·3785 ± ·0466 | ·3868 ± ·0303 |
| Rate of increase in weight | − ·0450 ± ·0457 | ·0745 ± ·0537 | ·1004 ± ·0346 | ·1237 | ·0356 | ·0691 | ·3375 ± ·0406 | ·4254 ± ·0442 | ·3843 ± ·0298 |
| Strength of grip ... ... | ·1675 ± ·0368 | ·0311 ± ·0481 | ·1601 ± ·0290 | ·0619 | − ·0288 | ·1288 | ·5770 ± ·0256 | ·5518 ± ·0335 | ·5732 ± ·0201 |
| Head circumference ... | ·0702 ± ·0378 | ·0568 ± ·0480 | ·0250 ± ·0298 | − ·0212 | ·0151 | − ·0174 | ·4349 ± ·0311 | ·4235 ± ·0395 | ·4242 ± ·0246 |
| GIRLS | With Thyroid Breadth corrected for age from *B1** growth curve | | | | | | Thyroid breadth corrected from *BN* curve* | With height corrected for age | With weight corrected for age |
| Factors corrected for age | Without correction for height | | | Corrected for height as above† | | | | | |
| | Category 01 | Categories 2+3 | All girls | Category 01 | Categories 2+3 | All girls | All girls | All girls | All girls |
| Height ... ... ... | ·2909 ± ·0485 | ·2251 ± ·0366 | ·2141 ± ·0298 | — | ·1561 | ·0993 | ·2222 ± ·0291 | — | ·7205 ± ·0148 |
| Weight ... ... ... | ·3413 ± ·0454 | ·3305 ± ·0344 | ·2427 ± ·0290 | ·1940 | ·4039 | ·2305 | ·2292 ± ·0292 | ·7205 ± ·0148 | — |
| Rate of growth ... ... | ·0134 ± ·0527 | ·1579 ± ·0389 | ·0254 ± ·0319 | − ·0598 | ·1455 | − ·0023 | ·0135 ± ·0319 | ·2373 ± ·0301 | — |
| Rate of increase in weight | ·2396 ± ·0498 | ·3695 ± ·0346 | ·1424 ± ·0313 | ·1436 | ·3718 | ·1062 | ·1272 ± ·0314 | ·3880 ± ·0291 | — |
| Strength of grip ... ... | ·2067 ± ·0480 | ·2058 ± ·0369 | ·1509 ± ·0299 | ·0730 | ·1972 | ·1075 | ·1265 ± ·0291 | ·5030 ± ·0229 | ·6376 ± ·0179 |
| Systolic pressure ... | ·0422 ± ·0590 | ·1660 ± ·0376 | ·1267 ± ·0301 | − ·0413 | ·1527 | ·0991 | ·1088 ± ·0293 | ·2757 ± ·0283 | — |
| Diastolic pressure ... | ·1340 ± ·0528 | ·1730 ± ·0397 | ·1427 ± ·0319 | ·1059 | ·1656 | ·1310 | ·1593 ± ·0307 | ·1149 ± ·0322 | — |
| Pulse pressure ... ... | ·0793 ± ·0535 | ·0259 ± ·0409 | ·0868 ± ·0323 | ·0317 | ·0140 | ·0684 | ·0363 ± ·0314 | ·1698 ± ·0317 | — |
| Pulse interval ... ... | ·1202 ± ·0494 | − ·0651 ± ·0384 | ·0588 ± ·0305 | ·1139 | ·0681 | ·0546 | ·0338 ± ·0296 | ·0391 ± ·0306 | — |
| Proficiency in class ... | ·1201 ± ·0504 | ·0820 ± ·0394 | ·0771 ± ·0312 | ·0413 | ·0641 | ·0766 | ·0718 ± ·0303 | ·2081 ± ·0300 | — |
| Diameter of neck ... | ·2220 ± ·0475 | ·6536 ± ·0336 | ·4279 ± ·0246 | ·0876 | ·6557 | ·4151 | — | ·5168‡ ± ·0368 | — |
| Circumference of neck ... | ·2989 ± ·0455 | ·5378 ± ·0450 | ·2951 ± ·0275 | ·1714 | ·5393 | ·2740 | — | ·5607‡ ± ·0346 | — |

* *B1* curve means the growth curve for Category 01 alone; *BN* is the slightly different growth curve deduced for the "non-goitrous" areas; see text. In boys the corrections do not differ sensibly.

† See text.

‡ Category 01 only.

and the coefficients for use in partial correlation are given by multiplying the regression coefficients above by the respective ratios of the standard deviations, giving finally the partial coefficients when the normal effect of height is eliminated from both measurements as follows:

| | Category 01 | Categories 2+3 | Complete sample |
|---|---|---|---|
| Thyroid breadth with neck diameter. | ·0876 | ·6557 | ·4151 |
| Thyroid breadth with neck circumference. | ·1714 | ·5393 | ·2740 |

The implications of these coefficients have been discussed in Section 2 (*c*).

(*f*) *Correlation with Horizontal Circumference of Head.* In the 540 boys this was measured by a steel tape. The actual means at each half-year age-group are shown in Table XXIII. After combining a few of these groups a freehand curve was drawn through the points and the smoothed values also shown in the same table were obtained. The growth curve, using the head circumference at age 11 as scale unit, is drawn in Figure 8, and it is evident that the relative rate of growth of the head in boys between ages 11 and 16 is very much the same as for the thyroid gland, both being notably less than for height.

After correction to age 12 the means and standard deviations were:

| | | |
|---|---|---|
| Category 01. | Mean 528·30 ± ·55, | $\sigma = 14{\cdot}4421$ mm. |
| Categories 2 + 3. | „ 527·61 ± ·75, | $\sigma = 15{\cdot}5787$ „ |
| Complete sample. | „ 527·90 ± ·43, | $\sigma = 14{\cdot}8918$ „ |

There is no significant difference between the means.

The correlation coefficients with thyroid breadth are given in Table XXVI; they are insignificant for all boys and also for each subgroup, and when the effect of height is eliminated the coefficients become practically zero. We must therefore conclude that there is no relation between thyroid size and size of head in boys apart from that to be normally expected on account of differing size of the boys at the same age. The correlation between head circumference and stature after age correction is of the order ·42.

### (6) Relation to body weight and physical strength.

(*a*) *Body weight.* The manner in which the weights of the children were obtained has been mentioned in Section (5), and the complete data are set out in Table XXII. These were fitted with cubic curves having the formulae:

$$\text{Boys} \quad W = 11{\cdot}145 + 1{\cdot}3127a - {\cdot}017752a^2 + {\cdot}005368a^3,$$

$$\text{Girls} \quad W = 12{\cdot}755 + {\cdot}4845a + {\cdot}05452a^2 + {\cdot}004272a^3,$$

where $W$ = weight in kgm., $a$ = age in years. Points on these curves at half-year intervals are given in Table XXIII and the growth curve with age for girls is

drawn in Figure 10, using the mean weight at age 11 exactly as scale unit. The cubic has been drawn as far as age 15, the rest of the curve being drawn freehand

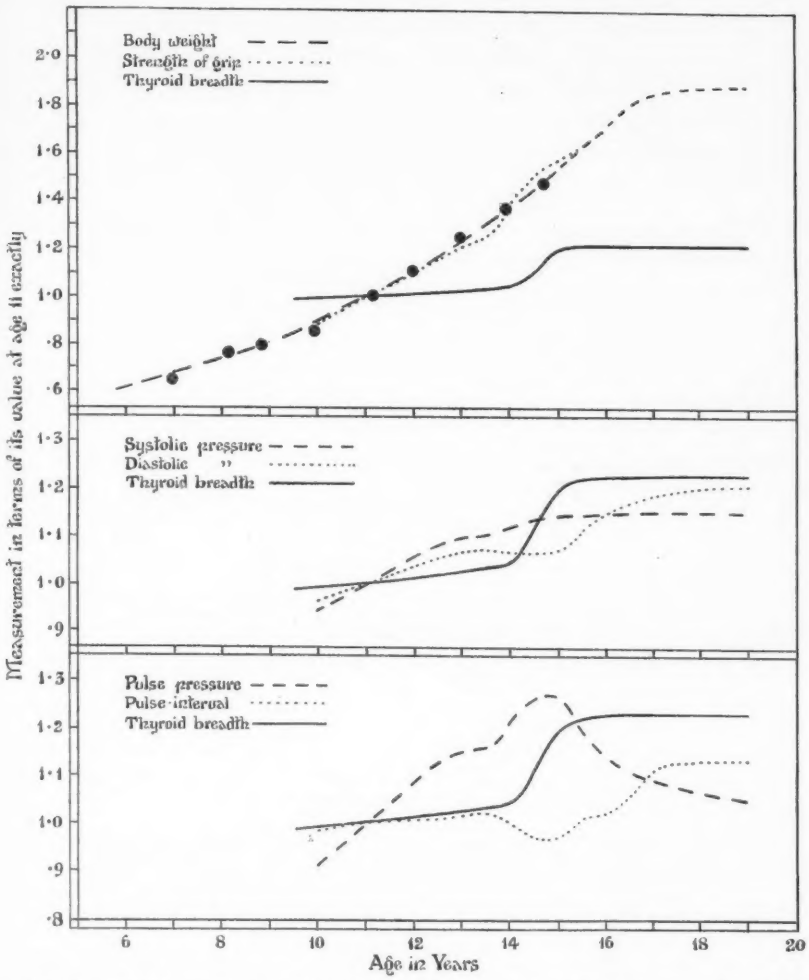



Fig. 10. Growth curves of thyroid gland in girls compared with growth curves for certain physical and physiological characters in the same districts, the mean measurement at age 11 exactly being taken as scale unit in each case.

on the basis of measurements of college students. The points inserted indicate that the cubic fits the data fairly well though possibly it fails to represent a very slight dip about age 10.

After correction to age 12 the means and standard deviations were:

| | | | | |
|---|---|---|---|---|
| Boys | Category 01. | Mean 33·06 ± ·21, | $\sigma = 5{\cdot}4520$ kgm. |
| | Categories 2 + 3. | „ 34·16 ± ·25, | 5·2046 „ |
| | Complete sample. | „ 33·49 ± ·16, | 5·3830 „ |
| Girls | Category 01. | Mean 33·06 ± ·31, | $\sigma = 6{\cdot}1819$ „ |
| | Categories 2 + 3. | „ 34·56 ± ·22, | 5·7465 „ |
| | Complete sample. | „ 33·18 ± ·19, | 6·2131 „ |

The correlation coefficients with thyroid breadth after age correction are shown in Table XXVI (p. 331), and there is at once obvious a contrast between the sexes. In normal boys and girls alike there is a correlation exceeding ·3; in boys this is entirely obliterated in the goitrous group and reduced to ·1187 in the complete sample, but in girls it is maintained in the goitrous group and reduced to ·24 in the complete sample. After correction for the effect of height by partial correlation as described in Section 5 (*e*) the coefficients remain significantly positive in the normal groups both for boys and girls; thus:

| | | |
|---|---|---|
| Category 01. | Boys $r =$ ·2394 ± ·0365, | Girls $r =$ ·1940 ± 0489. |
| Categories 2 + 3. | „ $r = -$ ·0863 ± ·0479, | „ $r =$ ·4039 ± ·0324. |
| Complete sample. | „ $r =$ ·0701 ± ·0301, | „ $r =$ ·2305 ± ·0292. |

In the goitrous group the correlation has become zero or slightly negative in boys but is positive to the extent of ·4 in girls; in the complete samples it has become insignificant in boys, but in girls there remains a significant relation of the order ·23.

The contrast is clearly shown by the regression lines of weight on thyroid breadth drawn in Figure 11; the data for these are given in Table XXIV and the means have been fitted with cubic and quartic curves as follows:

Boys $W' = \cdot 271 + 20{\cdot}114B' - 3{\cdot}8552B'^2 + {\cdot}23585B'^3$,

Girls $W' = -7{\cdot}906 + 27{\cdot}405B' - 6{\cdot}3928B'^2 + {\cdot}64386B'^3 - {\cdot}022952B'^4$,

where $W'$ and $B'$ are corrected to age 12 and the units are kgm. and cm. respectively. For boys $\eta = {\cdot}2136$, $\eta^2 - \overline{\eta}^2 = {\cdot}01956$, p.e. of $\overline{\eta}^2 = {\cdot}00890$, so $\eta$ is barely significant; for girls $\eta = {\cdot}2785$, $\eta^2 - \overline{\eta}^2 = {\cdot}07125$, p.e. of $\overline{\eta}^2 = {\cdot}00344$, so $\eta$ is undoubtedly significant.

On the same diagram the regression lines of thyroid breadth on weight are also drawn; these do not deviate sensibly from straight lines, and their equations are:

Boys $B' = 3{\cdot}8998 + {\cdot}0185\,(W' - 33{\cdot}504)$,

Girls $B' = 3{\cdot}5623 + {\cdot}0350\,(W' - 33{\cdot}278)$.

The conclusions which must be drawn are that (i) both in girls and boys thyroid glands which are larger within normal limits seem to be associated with body weights above average for the age in question even after correction for the height of the child, a result which would probably be obtained with any organ of soft consistency; (ii) in boys pronounced enlargement beyond normal limits has no positive relation with weight, the net result being that in a population of boys in a goitre district there is on the whole no significant tendency for goitres to occur

in the heavier boys at a given age; (iii) in girls the positive association between thyroid size and heaviness becomes much more pronounced in goitrous girls than normal girls after correcting for their ages and heights, and this certainly indicates that either ($\alpha$) the thyroid is more prone to enlarge in well-developed girls than in under-developed girls of the same age and height, or else ($\beta$) the gland in its

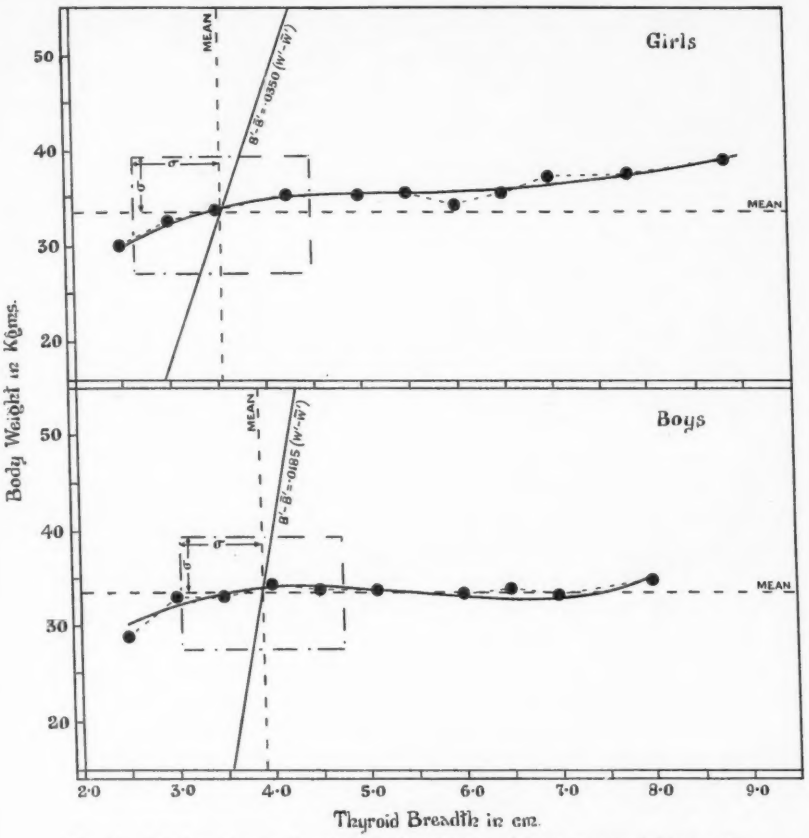



Fig. 11. Relations between thyroid breadth and body weight, both corrected to age 12.

enlarged state has a direct influence in raising the body weight above the average. The existence of a similar though slighter relation with height and further evidence in the following sections make it improbable that any explanation under the heading ($\beta$) is the correct one. The additional weight of the gland itself could have only a very slight effect in producing a positive correlation*.

* A rough calculation shows that this could not exceed ·002 in the normal group and ·005 in the other groups.

(*b*) *Strength of grip.* This was measured by means of a Mathieu's dynamometer allowing two grips with the right hand followed by two grips with the left. In 529 boys 325 had a stronger mean grip in the right than the left hand, 78 *vice versâ*, and in 126 the grips were equal; in 516 girls 381 were stronger in the right, 91 in the left and in 44 the grips were equal. The highest of the four readings has been used as the measure of strength of grip in the present work. The means of these maximum grips at half-year intervals are shown in Table XXIII for boys and girls, and these form a sufficiently smooth series after combining one or two age-groups to draw a freehand curve through without difficulty. Points on these curves are also shown in the same table. The curve for girls is drawn in Figure 10 using the grip at age 11 as scale unit, and it closely follows the curve of body weight excepting the presence of a rather more pronounced pubescent dip.

After correction to age 12 from these curves the means and standard deviations expressed in kilograms were as follows:

| | | | |
|---|---|---|---|
| Boys | Category 01. | Mean 18·595 ± ·152, | $\sigma = 4{\cdot}0029$. |
| | Categories 2 + 3. | „ 19·816 ± ·217, | 4·5080. |
| | Complete sample. | „ 19·089 ± ·129, | 4·2639. |
| Girls | Category 01. | Mean 16·500 ± ·203, | $\sigma = 4{\cdot}0465$. |
| | Categories 2 + 3. | „ 17·180 ± ·155, | 4·0312. |
| | Complete sample. | „ 16·598 ± ·129, | 4·0513. |

The correlation coefficients between grip and thyroid breadth after correction for age and also for height are shown in Table XXVI; the height-corrected values with their probable errors are:

| | | |
|---|---|---|
| Category 01. | Boys $r = {\cdot}0619 \pm {\cdot}0378$, | Girls $r = {\cdot}0730 \pm {\cdot}0499$. |
| Categories 2 + 3. | „ $r = -{\cdot}0288 \pm {\cdot}0481$, | „ $r = {\cdot}1972 \pm {\cdot}0371$. |
| Complete sample. | „ $r = {\cdot}1288 \pm {\cdot}0297$, | „ $r = {\cdot}1075 \pm {\cdot}0302$. |

These indicate again an important sex difference; in boys and girls alike the thyroid size within normal limits is positively related to strength of grip in children of the same age, but this relation becomes insignificant when corrected for height as well; in boys the relation is obliterated in the goitre group alone but in girls a pronounced correlation with grip remains, indicating some subtle relation between proneness to goitrous enlargement and the state of physical development.

This might be a secondary result of the association just noticed with body weight; thus in both sexes there is a correlation between grip and weight, even after correction for height, as shown by the values below all corrected for age:

| | Boys | Girls |
|---|---|---|
| Strength of grip with weight. | $r = {\cdot}6376 \pm {\cdot}0179$, | ·5936 ± ·0199. |
| „ „ height. | $r = {\cdot}5732 \pm {\cdot}0201$, | ·5030 ± ·0229. |
| Weight with height. | $r = {\cdot}8069 \pm {\cdot}0105$, | ·7205 ± ·0148. |
| Grip with weight, height constant. | $r = {\cdot}3618 \pm {\cdot}0262$, | ·3859 ± ·0262. |

After partial correlation, making age, height and weight all constant the correlation for the goitrous group of girls between thyroid size and grip becomes $r = \cdot0489 \pm \cdot0384$, and this seems to show that the relation between thyroid size and grip is secondary to the relation with general physical development. Proceeding in the same way with the correlation coefficients for the complete samples:

| | Boys | Girls |
|---|---|---|
| Thyroid breadth with grip, age, height and weight constant | $r = \cdot1112 \pm \cdot0298$, | $\cdot0206 \pm \cdot0302$. |

It may be concluded that all the relations between thyroid size and grip are due to general physical development; but in boys there may be a residual relationship not so explained which is confined to the slighter degrees of thyroid enlargement.

It is of interest to note that in girls, when age, height and grip are all made constant, the correlation coefficients between thyroid breadth and weight still remain considerable; thus:

Category 01, $r = \cdot1705$; Categories 2 + 3, $r = \cdot3848$; Complete sample, $r = \cdot1950$.

From this it appears likely that the relation of the thyroid size with body weight is only to a small extent a relation with *muscular* development.

The regression of grip on thyroid breadth is shown in Table XXIV.

### (7) Relation to rate of growth.

This has been investigated by two different methods applied both to rates of increase in height and weight.

(*a*) Wherever possible the past records of height and weight of every child were extracted giving the age distributions of Table XXII. By separating the Cheshire girls into three thyroid categories and fitting cubic curves to the mean heights and weights of each group, as far back as age 3 or 4 in some cases, the following curves were obtained:

$$\text{Category 01.}\quad H_1 = 73{\cdot}238 + 5{\cdot}0991a + {\cdot}11973a^2 - {\cdot}006594a^3,$$
$$\text{„ 2.}\quad H_2 = 64{\cdot}398 + 9{\cdot}4739a - {\cdot}41182a^2 + {\cdot}013151a^3,$$
$$\text{„ 3.}\quad H_3 = 87{\cdot}721 + 1{\cdot}3962a + {\cdot}48515a^2 - {\cdot}017477a^3,$$
$$\text{Category 01.}\quad W_1 = 12{\cdot}349 + {\cdot}5791a + {\cdot}031810a^2 + {\cdot}004932a^3,$$
$$\text{„ 2.}\quad W_2 = 10{\cdot}551 + 1{\cdot}4179a - {\cdot}053196a^2 + {\cdot}008260a^3,$$
$$\text{„ 3.}\quad W_3 = 13{\cdot}109 + {\cdot}1386a + {\cdot}122366a^2 + {\cdot}001840a^3,$$

—the numbers of observations forming these curves being respectively 481, 709 and 211 for height, and 481, 708 and 206 for weight.

It would be unsafe to place much reliance on these curves outside the limits of ages 8 to 14, but within those limits they should represent with fair accuracy the average manner of growth of girls (1) who do not develop any enlargement of the gland during the period, (2) who develop a moderate degree of enlargement, (3) who develop pronounced goitres. By differentiating with respect to age, $a$, the proportionate annual rates of growth $\frac{1}{H} \cdot \frac{dH}{da}$ and $\frac{1}{W} \cdot \frac{dW}{da}$ can be found at each year of age, and also the ratios of weight to height, $W/H$, for each group.

The conclusions suggested by Table XXVII are that:

(i) girls who exhibit *slight* thyroid enlargements before 14 are characterised by a rate of growth in height below average in height, and an average rate of growth in weight, from 8 to 11, compensated for about puberty by an unusually rapid growth both in height and weight after the 12th year has been reached; their weight per cm. of height is therefore higher than in the normal group of girls at all ages from 8 to 14;

TABLE XXVII.

*Girls.*

| Age | *8* | *9* | *10* | *11* | *12* | *13* | *14* | Mean Values | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | *8—11* | *12—14* |
| $\frac{1}{H_1}\cdot\frac{dH_1}{da}$ | ·0486 | ·0456 | ·0426 | ·0395 | ·0365 | ·0335 | ·0305 | ·0441 | ·0335 |
| $\frac{1}{H_2}\cdot\frac{dH_2}{da}$ | ·0449 | ·0418 | ·0395 | ·0381 | ·0373 | ·0370 | ·0345 | ·0411 | ·0372 |
| $\frac{1}{H_3}\cdot\frac{dH_3}{da}$ | ·0480 | ·0464 | ·0441 | ·0413 | ·0381 | ·0345 | ·0305 | ·0449 | ·0344 |
| $\frac{1}{W_1}\cdot\frac{dW_1}{da}$ | ·0945 | ·0990 | ·1027 | ·1054 | ·1072 | ·1083 | ·1086 | ·1004 | ·1037 |
| $\frac{1}{W_2}\cdot\frac{dW_2}{da}$ | ·0948 | ·0986 | ·1023 | ·1057 | ·1085 | ·1107 | ·1122 | ·1003 | ·1105 |
| $\frac{1}{W_3}\cdot\frac{dW_3}{da}$ | ·1066 | ·1089 | ·1098 | ·1097 | ·1088 | ·1073 | ·1054 | ·1087 | ·1072 |
| $\frac{W_1}{H_1}$ | ·1821 | ·1914 | ·2026 | ·2157 | ·2310 | ·2484 | ·2682 | ·1979 | ·2492 |
| $\frac{W_2}{H_2}$ | ·1884 | ·1988 | ·2111 | ·2253 | ·2415 | ·2597 | ·2798 | ·2059 | ·2603 |
| $\frac{W_3}{H_3}$ | ·1900 | ·2019 | ·2153 | ·2302 | ·2468 | ·2652 | ·2855 | ·2093 | ·2658 |

(ii) girls who develop *large* goitres are characterised by rates of growth above average both in height and weight from the 8th to about the 13th year, compensated for afterwards by a less rapid growth; their weight per cm. of stature is higher than for either the normal or Category 2 groups of girls at every age.

How reliable these indications may be there is no satisfactory method of judging; the number of measurements used for the curves was, however, considerable*. The question is further discussed in subsection (*b*).

* Conclusions (i) and (ii) above have been also tested by drawing freehand curves through the points instead of fitting cubics and it is found that they still hold good, the only appreciable modifications being that the value of $\frac{1}{H_1}\cdot\frac{dH_1}{da}$ would be smaller at age 9 and correspondingly larger at age 10, and the values of $\frac{1}{W_1}\cdot\frac{dW_1}{da}$ and $\frac{1}{W_2}\cdot\frac{dW_2}{da}$ would be somewhat larger at age 12 and smaller at 13 than the values in Table XXVII. It must of course be remembered that whatever method of curve fitting be used, individual peculiarities of growth associated with puberty tend to disappear in mean curves for groups of children owing to the variation in chronological age at which puberty occurs.

The indices of weight per cm. of height were worked out also for the boys, using two groups only and taking the ratios of the means without smoothing.

TABLE XXVIII.

*Boys.*

| Age | *8* | *9* | *10* | *11* | *12* | *13* | *14* | *8—11* | *12—14* |
|---|---|---|---|---|---|---|---|---|---|
| $W_1/H_1$ ... | ·1937 | ·2030 | ·2138 | ·2250 | ·2400 | ·2490 | ·2713 | ·2039 | ·2534 |
| $W_{23}/H_{23}$ ... | ·1965 | ·2040 | ·2151 | ·2273 | ·2429 | ·2553 | ·2686 | ·2107 | ·2556 |

In this case there is again an excess in the ratios for the goitre group at almost every age, but not so pronounced as in the case of the girls.

(*b*) The problem has been attacked by the correlation method, taking into account only the rates of growth during the year or two immediately preceding the measurement of the thyroid. In order to correct this for age the following method was used.

The present (or most recent) height was corrected to age 12 by means of the cubic curve obtained in Section (5) giving a value $H_0$; the record preceding this at an interval of not less than a year before was similarly corrected from the age at which it was taken to age 12, giving a value $H_0'$. If growth had been proceeding in the interval between these measurements at the rate given by the cubic curve we should have $H_0 = H_0'$; if growth had been more rapid than the cubic curve, $H_0 > H_0'$, if less rapid then $H_0 < H_0'$. If $t$ = interval in years between the dates of the measurements, the rate of growth in cm. per year in excess or defect of the mean rate during the age period dealt with is given by $h = \dfrac{H_0 - H_0'}{t}$, which may have a + or − sign.

The same method was used for increase in weight giving rates of growth in excess or defect of the mean rate during the age period covered, $\omega = \dfrac{W_0 - W_0'}{t}$. The values of $h$ and $\omega$ were then correlated with thyroid breadth corrected for age. The mean rates after combining the categories in their correct proportions being slightly in excess of zero, the whole series of means given in Table XXIV and used in drawing the regression lines in Figure 12 have been adjusted by subtracting this small deviation to bring the mean rate for the whole to exactly zero.

The standard deviations were as follows:

| | | | |
|---|---|---|---|
| Boys | Category 01. | $\sigma_h = 2{\cdot}7773$, | $\sigma_\omega = 1{\cdot}8862$. |
| | Categories 2 + 3. | $\sigma_h = 2{\cdot}6073$, | $\sigma_\omega = 1{\cdot}8827$. |
| | Complete sample. | $\sigma_h = 2{\cdot}7066$, | $\sigma_\omega = 1{\cdot}8970$. |
| Girls | Category 01. | $\sigma_h = 1{\cdot}9392$, | $\sigma_\omega = 1{\cdot}8027$. |
| | Categories 2 + 3. | $\sigma_h = 1{\cdot}5865$, | $\sigma_\omega = 1{\cdot}4831$. |
| | Complete sample. | $\sigma_h = 1{\cdot}8979$, | $\sigma_\omega = 1{\cdot}7588$. |

It is noteworthy that the standard deviations are considerably smaller for $h$ and $\omega$ in the goitre groups of girls than in the normal group, but there is little difference for boys.

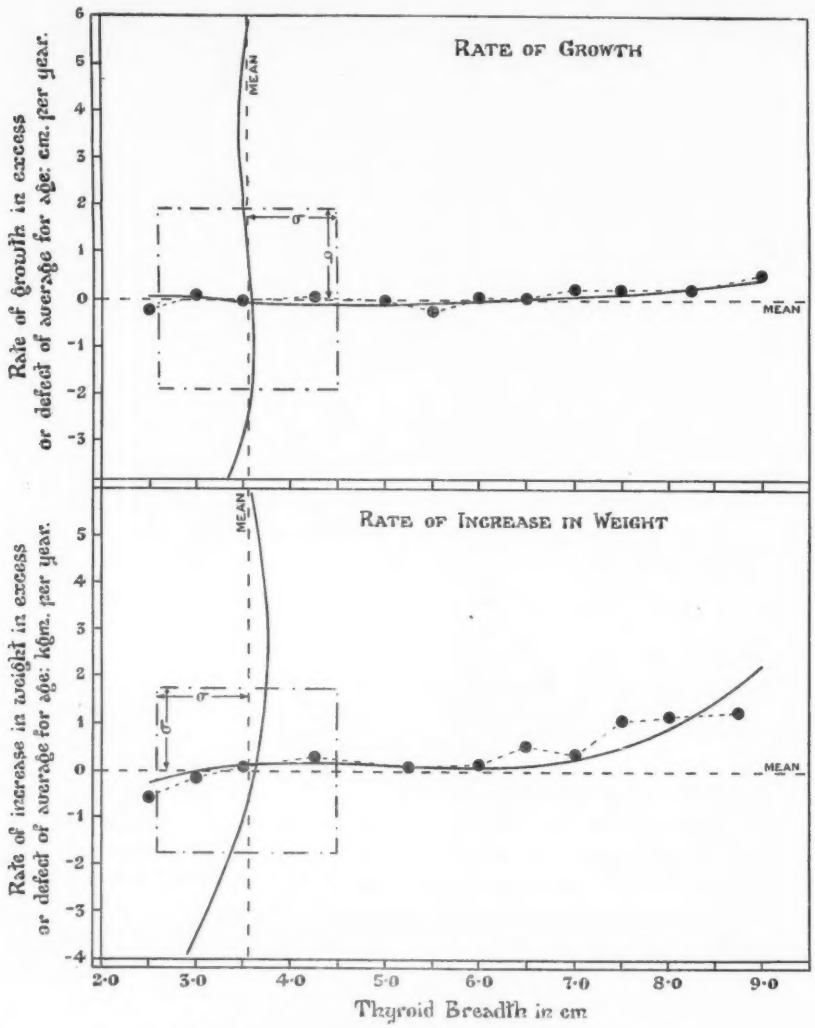



Fig. 12. Relations between thyroid breadth and rates of growth in Cheshire girls, both corrected to age 12.

The correlation coefficients with thyroid breadth corrected for age, and the partial coefficients after correction for height as before, are given in Table XXVI; the partial coefficients with their probable errors were:

| | | | |
|---|---|---|---|
| Boys | Category 01. | $r_{Bh.H} = -\cdot0336 \pm \cdot0471$, | $r_{B\omega.H} = \cdot1237 \pm \cdot0451$. |
| | Categories 2 + 3. | $-\cdot0852 \pm \cdot0540$, | $\cdot0356 \pm \cdot0539$. |
| | Complete sample. | $-\cdot0212 \pm \cdot0356$, | $\cdot0691 \pm \cdot0348$. |
| Girls | Category 01. | $r_{Bh.H} = -\cdot0598 \pm \cdot0525$, | $r_{B\omega.H} = \cdot1436 \pm \cdot0517$. |
| | Categories 2 + 3. | $\cdot1455 \pm \cdot0392$, | $\cdot3718 \pm \cdot0345$. |
| | Complete sample. | $-\cdot0023 \pm \cdot0319$, | $\cdot1062 \pm \cdot0316$. |

The coefficients in boys all become insignificant when the normal effect of size has been corrected for, except a small residual coefficient in the normal group between rate of increase in weight and thyroid breadth; in boys goitrous enlargement is not therefore associated to any measurable extent with rapid growth.

In girls, however, there remains a significant relation between rate of growth in height and thyroid size in the goitre group, though not in the normal or complete groups, indicating that among girls of the same age and height with goitres, the larger goitres tend to occur on the whole in the girls who have recently been growing quickly; with rate of increase in weight there remains a significant correlation in both normal and goitre groups, but this is much more pronounced in the goitre group. These relations are made somewhat clearer by the regression lines in Figure 12; cubic curves have been fitted to all the four regressions, their equations being:

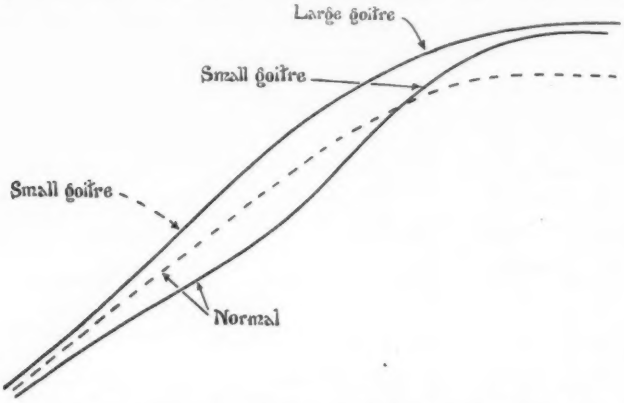
$$h \text{ on } B' \ldots h = \cdot3895 - \cdot16030B' + \cdot006552B'^2 + \cdot0013606B'^3,$$
$$B' \text{ on } h \ldots B' = 3\cdot5821 - \cdot02400h - \cdot011481h^2 + \cdot002580h^3,$$
$$\omega \text{ on } B' \ldots \omega = -4\cdot1263 + 2\cdot71267B' + \cdot560847B'^2 + \cdot037562B'^3,$$
$$B' \text{ on } \omega \ldots B' = 3\cdot6111 + \cdot10099\omega - \cdot020691\omega^2 + \cdot000520\omega^3.$$

Examination of the upper curves in Figure 12 shows a slight tendency for rate of growth in height to be more rapid than the average in girls whose thyroids exceed 5 or 6 cm. in breadth; there seems, however, to be no tendency for girls growing at rates above the average for their age to have on the whole larger thyroids. The lower set of curves shows that very small thyroids are associated with slower growth in weight, and goitres exceeding about 6 cm. in breadth are associated with a notably high rate of growth; moderate degrees of enlargement have no relation with rate of increase in weight. There is also a very definite tendency for girls gaining weight at a rate below average to have small thyroids, an effect which would probably be found for most organs of the body, but little or no tendency is indicated for the thyroid to go on increasing as the rate of growth in weight is more and more excessive.

This result would seem paradoxical were it not that the rate of growth we are here considering is limited to a short period immediately preceding measurement of the gland. Since goitres usually take several years to reach a large size we should expect that a moderately excessive rate of growth dating back over several

years would be more characteristic of the large goitre group than a very excessive rate of growth dating back for only a short time, and therefore since very high rates of growth are of short duration and usually compensated to some extent by a period of slow growth, we should expect those girls who had recently been growing very rapidly to be characterised by small rather than large goitres; in other words that the regression of thyroid size on rate of growth would become zero or negative as the rate of growth reached high values. This is just what is indicated by the regression lines and the results in subsection ($\alpha$) above; moreover it is not incompatible with the regression lines of Figure 10 which indicated a linear relation with *present* height and weight.

It seems therefore that large goitres are more likely to arise in a goitre district in those girls whose rate of growth in height and general development is above the average over a continuous period of years whilst small enlargements are more




likely to be found in those girls whose growth is at first slow and then very rapid for a short period, bringing both groups eventually above the average for their age. This tendency has been represented in diagrammatic form. A comparison of this diagram with the analysed curves of blood pressure for girls with large, small and zero goitres in Figure 14 shows a remarkable similarity, which strengthens these conclusions.

In a previous paper(1) it was concluded that girls in Bern where goitre is severely endemic exhibit a retardation in growth increasing with the size of the goitre, which retardation is counteracted by regular iodine administration. At first sight this effect seems contradictory to the relation found in English girls and this might suggest that the forms of goitre in the two countries are different in character. The present research gives no evidence in support of such an idea but strongly suggests that they only differ as regards the intensity of the factor producing them. No theory of infection alone is capable of explaining the facts on either assumption. The iodine-deprivation theory of causation is however consistent with all the facts, difficult as they seem at first sight; thus in a district

such as Cheshire where there must be a *slight* deficiency in iodine intake on this theory, symptoms of deficiency will chiefly appear, *ceteris paribus*, in those girls needing most thyroid secretion to maintain their growth—namely in the best developed and quickly growing girls—hence we find positive correlations of thyroid size with growth factors due to this selection which obscure any slight retardation in growth which may result in these bigger girls. Where however the deprivation becomes so *severe* and universal that practically all girls whether quickly or slowly growing are affected, as at Bern, the correlation due to selection disappears and the opposite effect due to retardation becomes evident.

### (8) RELATION TO BLOOD PRESSURE AND PULSE RATE.

The blood pressures and pulse rates of all the girls were measured in the sitting posture, using the auscultatory method checked by palpation. The index for the diastolic pressure was the point at which sudden dulling of the sound from "clear" to "muffled" occurs, usually termed the 4th point or end of the 3rd phase. The same instrument was used as in a similar work on boys, the results of which were published in 1924(8); this instrument was a Brunton sphygmomanometer with 12 cm. cuff; it had previously been tested as to accuracy against a mercury manometer.

One of the reasons for measuring blood pressures in the present research was that correlations have been found between basal metabolic rate and pulse pressure

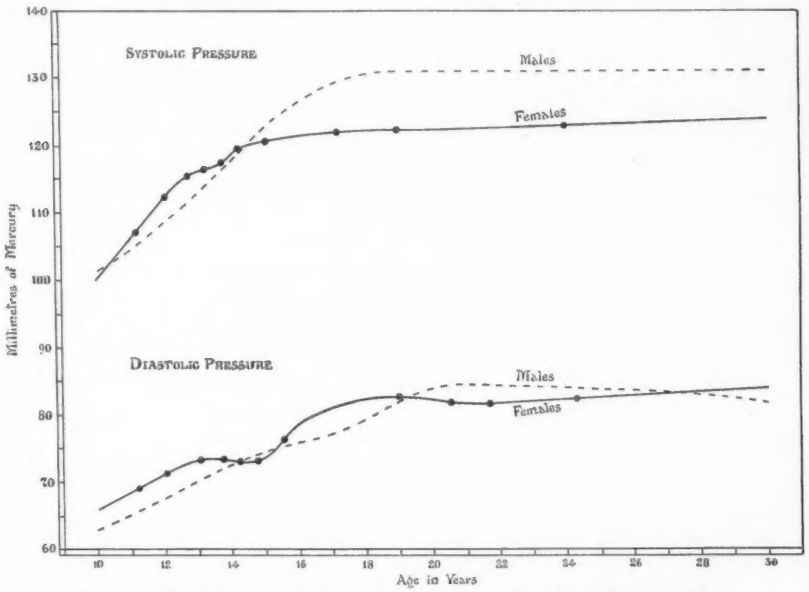



Fig. 13. Effect of age on blood pressure. Cheshire and London girls and students (compared with London boys and students).

as well as pulse rate by Read and others, and though this may be only characteristic of exophthalmic goitre, it suggested that a relation of some kind might be found between pulse pressure and the size of the thyroid in adolescence.

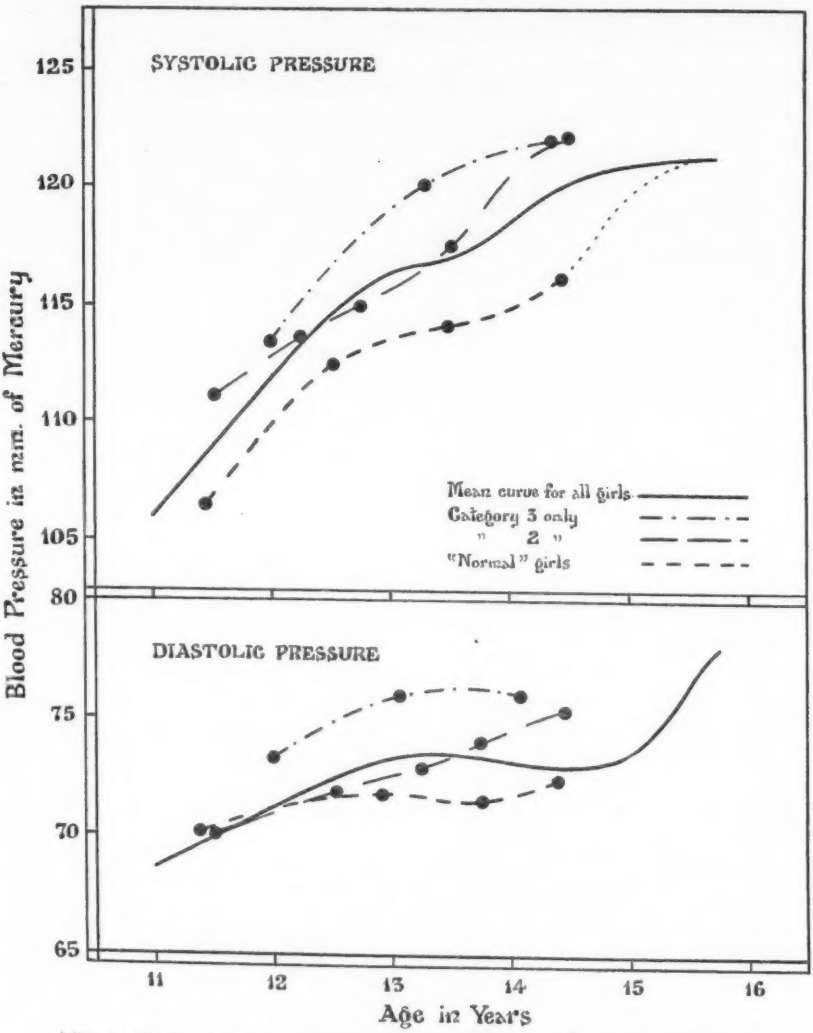



Fig. 14. Effect of age on blood pressure in goitrous and non-goitrous girls (Cheshire).

The relation of blood pressure to age has been worked out for boys in the memoir referred to above (8), but no extended series of blood pressure data in girls

has been hitherto published from the Galton Laboratory. In constructing the age curves shown in Figure 13 the data from the present study have been combined with blood pressure measurements carried out by an exactly similar technique with the same instrument on 326 women students of University College between the ages of 16 and 30. It is not proposed in the present paper to make any analysis of this data, and the details are not therefore given; the mean pressures at each age for the 537 school girls are shown in Table XXIII, and these and the students' data have been combined in producing the curves of Figure 13 which were readily drawn after combining a few neighbouring age-groups.

A comparison with the similar curves for males in Figures 5 and 6 of memoir (8), which have been reproduced as dotted lines in Figure 13, reveals interesting differences. The systolic pressure in girls is about the same as in boys at the 10th year but rises then more rapidly than in boys, almost reaching its adult level at 15, or some 3 years earlier than in boys; there is also in girls a pronounced pubescent dip about the 13th year. The diastolic pressure is some 3 mm. higher in girls than boys from 10 to 13, but a remarkable pubescent dip from 13 to 15 brings it below the boys' level; it then rises rapidly, reaching the adult level about 17, again about 3 years ahead of the boys in whom a similar rapid rise occurs from 17 to 20. In Figure 10 these curves are reduced to a scale unit equal to the pressure at age 11 in each case for comparison with the development of the thyroid gland on the same scale; this brings out a striking similarity between the curves for thyroid breadth and diastolic pressure, both showing a pronounced retardation about the period 12 to 14 followed by a rapid increase which is not however quite synchronous for the two factors. In Figure 14 the curves for normal girls have been separated from the composite data, and it then appears that in girls showing no visible enlargement the rapid rise in diastolic pressure begins at the end of the 14th year and is therefore synchronous with the rapid increase in normal thyroid size. In the lower part of Figure 10 the curve for pulse pressure has the same remarkable shape as was found in boys (memoir (8), Fig. 7), but the maximal peak is at 14—15 instead of 17—18 as in boys. The mean pulse pressures up to age 16 are shown in Table XXIII.

The mean pressures according to the smoothed curves from ages 10 to 20 in girls are given below in Table XXIX, and for comparison the corresponding values for the boys' curves referred to.

TABLE XXIX.

| Age | | *10* | *11* | *12* | *13* | *14* | *15* | *16* | *17* | *18* | *19* | *20* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Systolic | Girls ... | 100·0 | 106·0 | 112·0 | 116·3 | 118·4 | 120·8 | 121·5 | 121·9 | 122·1 | 122·2 | 122·4 |
| | *Boys* ... | *101* | *104* | *109* | *113* | *118* | *123* | *127* | *129* | *131* | *131* | *131* |
| Diastolic | Girls ... | 66·0 | 68·6 | 71·3 | 73·3 | 73·2 | 73·6 | 78·9 | 81·1 | 82·3 | 82·8 | 82·2 |
| | *Boys* ... | *63* | *65* | *68* | *70* | *73* | *75* | *76* | *77* | *79* | *82* | *84* |
| Pulse pressure | Girls ... | 34·0 | 37·4 | 40·7 | 43·0 | 45·2 | 47·2 | 42·6 | 40·8 | 39·8 | 39·4 | 40·2 |
| | *Boys* ... | *38* | *39* | *41* | *43* | *45* | *48* | *51* | *52* | *52* | *49* | *47* |

The mean intervals between pulse beats at half-year periods are shown in Table XXIII. There is no very consistent change with age between 10 and 15 years though a slight slowing of the pulse is perhaps indicated up to the 13th year followed by a quickening from 13½ to 15. Continuation of the curve by means of the data for women students seems to indicate a slowing of the pulse from the 15th to 17th year where it attains the adult level. The complete curve reduced to a scale with the value at 11 years as unit is shown in Figure 10 and it is seen to be somewhat complementary to the pulse pressure curve. It is noteworthy that the rapid increase in average mean size of the thyroid from 13½ to 15 years is accompanied by a synchronous but less pronounced shortening of the mean pulse interval or quickening of pulse rate, which seems to be a temporary phenomenon only. There is no doubt that the changes manifested in all the curves in Figure 10 between 13½ and 15 years are directly associated with the establishment of menstruation.

In Figure 14 separate curves of systolic and diastolic pressure with age have been drawn in a somewhat rough manner through the points indicated, which are the mean pressures of the three categories of Cheshire girls after combining a few neighbouring half-year groups. The probable errors of individual points are of course considerable, but the manner in which the curves for the large and small goitre groups conform to the conclusions regarding growth at the end of the last section can scarcely be fortuitous.

The means and standard deviations after correction to age 12 were as follows:

TABLE XXX.

| | | Category 01 | Categories 2+3 | Complete sample |
|---|---|---|---|---|
| Systolic pressure, mm. of mercury | Mean ... | 109·94 ± ·60 | 111·40 ± ·43 | 110·45 ± ·36 |
| | $\sigma$ ... | 12·067 | 12·008 | 11·625 |
| Diastolic pressure, mm. of mercury | Mean ... | 71·05 ± ·39 | 72·58 ± ·36 | 71·27 ± ·25 |
| | $\sigma$ ... | 7·343 | 8·850 | 7·569 |
| Pulse pressure, mm. of mercury | Mean ... | 40·76 ± ·60 | 43·11 ± ·42 | 41·11 ± ·36 |
| | $\sigma$ | 11·131 | 10·239 | 11·045 |
| Pulse interval, seconds | Mean ... | ·668 ± ·005 | ·680 ± ·004 | ·670 ± ·003 |
| | $\sigma$ ... | ·1065 | ·1144 | ·0951 |

The mean systolic pressures above differ from the sums of the mean diastolic and pulse pressures owing to the fact that in 12 °/₀ of the girls the diastolic pressure could not be read with certainty, so the total systolic readings were in excess of those of diastolic or of pulse pressures. After correction to age 12 by the curves, the correlation coefficients with thyroid breadth were as shown in Table XXVI. After further correction for the normal effect of height by partial correlation as explained in Section (5) these become

TABLE XXXI.

| Thyroid breadth with | Category 01 | Categories 2+3 | Complete sample |
|---|---|---|---|
| Systolic pressure ... | $-\cdot0413\pm\cdot0500$ | $\cdot1527\pm\cdot0377$ | $\cdot0991\pm\cdot0303$ |
| Diastolic „ ... | $\cdot1059\pm\cdot0532$ | $\cdot1656\pm\cdot0398$ | $\cdot1310\pm\cdot0320$ |
| Pulse „ ... | $\cdot0317\pm\cdot0538$ | $\cdot0140\pm\cdot0409$ | $\cdot0684\pm\cdot0324$ |
| Pulse interval ... | $\cdot1139\pm\cdot0495$ | $\cdot0681\pm\cdot0384$ | $\cdot0546\pm\cdot0305$ |

Systolic pressure has a significant relation with thyroid size in the goitre group and the complete group, but not amongst normal girls alone. The regression line on thyroid breadth has been represented in Figure 15 by a cubic curve with the formula

$$p' = 112\cdot542 - 3\cdot2879B' + \cdot93206B'^2 - \cdot058180B'^3,$$

where $p'$ = systolic pressure in mm. of mercury corrected to age 12, $B'$ = thyroid breadth in cm. also corrected for age. The actual means and frequencies are given in Table XXIV.

Diastolic pressure has significant correlations with thyroid breadth in all groups after correction for height. The regression line shown in Figure 15 has been fitted with a cubic curve having the formula

$$p' = 55\cdot730 + 8\cdot1649B' - 1\cdot41097B'^2 + \cdot086324B'^3,$$

the actual means being shown in Table XXIV.

Pulse pressure has no significant relation with thyroid size either amongst the normal or goitrous girls.

The conclusion is that enlargement of the thyroid about the period of puberty is associated to a slight but significant extent with raised diastolic pressure whatever the degree of enlargement may be and after correction for age and height; when the enlargement becomes pronounced it becomes also associated with a slight rise of mean systolic pressure, the pulse pressure being therefore not affected.

Pulse interval has a smaller positive relation with thyroid size which is significant amongst normal girls but insignificant in the other groups; this indicates a slightly slower pulse in girls with larger thyroids. The mean points have been fitted with a cubic curve having the formula

$$\text{P.I.} = \cdot5433 + \cdot06495B' - \cdot009168B'^2 + \cdot0003743B'^3,$$

which is drawn in Figure 14. The points are somewhat irregular and would be almost as well fitted by the straight line

$$\text{P.I.} = \cdot6702 + \cdot00389\,(B' - 3\cdot533).$$

There is a suggestion from the mean points that thyroids enlarged to the extent of 6—8 cm. were associated with a more rapid pulse rate; but this certainly does not tend to occur with slight or moderate degrees of enlargement,

in fact there is a slight slowing of pulse rate as indicated by Figure 15 and by the significant correlation with pulse interval in the normal group. This, coupled with

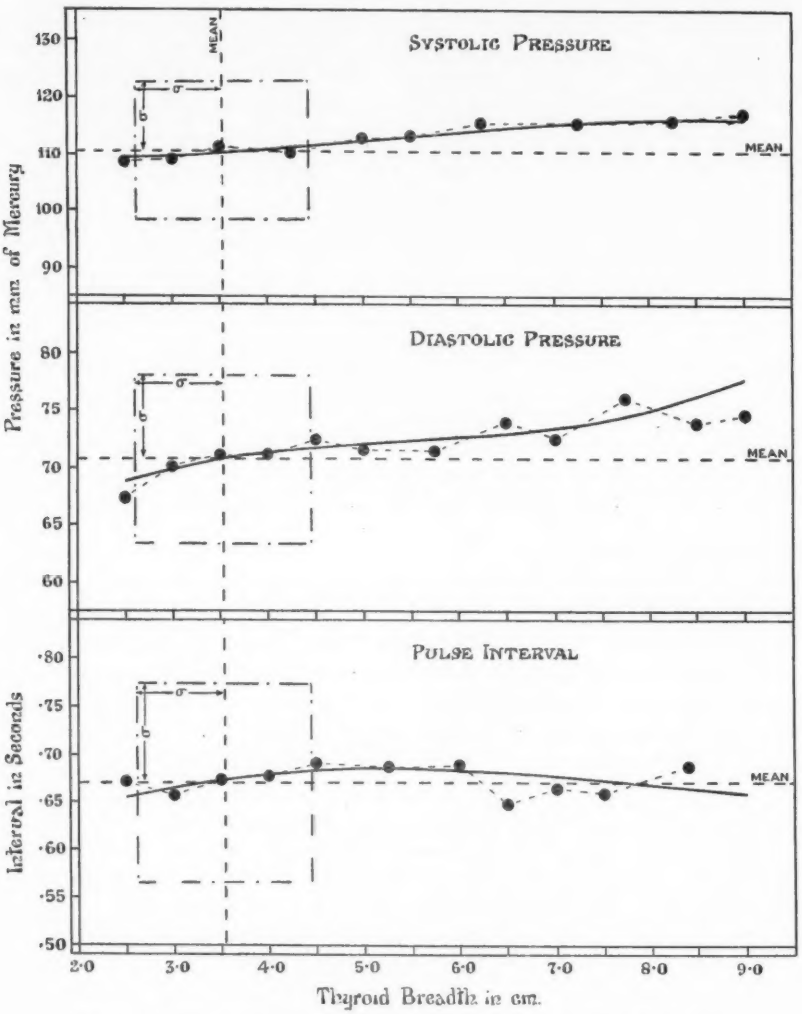



Fig. 15. Regression of blood pressure and pulse interval on thyroid breadth in girls: all corrected to age 12.

the absence of correlation with pulse pressure, seems to definitely rule out the presence of hyperthyroidism to any extent in the slight or moderate degrees of

enlargement met with about puberty, and to indicate rather the presence of factors tending to depress the pulse rate.

This again supports the view that the so-called "physiological" enlargements of the gland in girls are a manifestation of an effort by the gland, only partially successful, to overcome a deficiency of some substance stored by the gland and necessary to maintain growth—and that in this respect they are not essentially different in causation from the larger variety of goitre. That a large or small goitre may by the action of some nervous factor become complicated in some cases and lead to Graves' disease no doubt accounts for the presence of symptoms of hyperthyroidism in a certain proportion of girls with large goitres, but this is not evidence that the factors responsible for producing the goitre in the first instance were not of the same nature in all cases.

### (9) RELATION TO COLOUR OF EYES AND HAIR.

The colour of the eyes and hair was recorded for every girl in order to determine whether these racial characters had any association with tendency to goitre. The *eye* scale used was Martin's and the 16 shades of diminishing brown pigment yielded the following frequencies, these girls being almost entirely drawn from rural schools in north Cheshire.

| Eye shade number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Frequency ... | — | 3 | 9 | 41 | 30 | 28 | 34 | 29 | 13 | 44 | 45 | 13 | 51 | 66 | 52 | 24 |

For correlation purposes an arbitrary division was made into five groups, viz.: Very Dark = 1—4, Dark = 5—8, Medium = 9—12, Light = 13—14, Very Light = 15—16.

The mean thyroid breadths after age correction were

| V. Dark | Dark | Medium | Light | V. Light |
|---|---|---|---|---|
| 3·763 ± ·080 | 3·653 ± ·054 | 3·651 ± ·056 | 3·645 ± ·054 | 3·617 ± ·068. |

These do not differ significantly. The correlation coefficient obtained by treating the groups as parts of a normal frequency distribution was with class index correction $r = ·0345 \pm ·0303$, where the + sign indicates more pigment. The corrected value of $\eta$ is ·0754, $\eta^2 - \overline{\eta}^2 = ·003291$ and p.e. of $\overline{\eta}^2 = ·00186$. There is therefore no significant relation between thyroid size and colour of the eyes.

The *hair* colour was classified as Dark, Medium, Light, Very Light and Red. The mean thyroid breadths corrected for age and the actual numbers of girls in the groups were

| | Dark | Medium | Light and V. Light | Red |
|---|---|---|---|---|
| Mean breadth | 3·695 ± ·064 | 3·671 ± ·037 | 3·564 ± ·055 | 4·136 ± ·139 |
| No. of girls | 91 | 262 | 120 | 19 |

There are no significant differences between the first three groups, but the small group of red-haired girls gives a high value for the mean thyroid breadth

which differs significantly from that of the light-haired group, suggesting that red-haired girls are more liable to goitre: the number in this group however is small. The correlation coefficient obtained by the triserial method, combining the last two groups, is with class index correction $r = -\cdot0140 \pm \cdot0304$, the + sign indicating darkness; the corrected value of $\eta$ is $\cdot1068$, $\eta^2 - \bar{\eta}^2 = \cdot001433$, p.e. of $\bar{\eta}^2 = \cdot0016$. There is therefore no significant relation between darkness of hair and thyroid size.

### (10) Relation to proficiency in school work.

This was assessed by the following method. The Head Teachers were asked to assign to each girl her rank, as based upon class lists or the Teacher's opinion, amongst all the girls in her class who were born in the same year, specifying the number of such girls. Where girls born in a given year were drawn from more than one class, it was assumed that the highest girl in the lower class followed on in order below the lowest girl in the higher class born in the same year. This assumption is admittedly open to objections since actually there would be an overlap, but in general the errors introduced by this assumption would be small. If the number of girls born in the year 1914 in the upper three classes of a school were $n_1$, $n_2$, $n_3$, and a girl born in 1914 was ranked 4th in the 3rd class amongst the $n_3$ girls of the same age, her proficiency would be measured by the fraction $\frac{n_1 + n_2 + 4}{n_1 + n_2 + n_3}$; if she was placed 5th in the 1st class amongst the $n_1$ girls, her proficiency would be given by $\frac{5}{n_1 + n_2 + n_3}$.

The fractions thus obtained were reduced to give the place in a class of 100 girls born in the same year. Thus if the girl's rank be $x$ amongst $n$ girls, her rank in 100 was taken to be $100 \times \frac{x - \frac{1}{2}}{n}$ to the nearest integer; this amounts to placing her at the centre of the $x$th interval of uniform size $\frac{100}{n}$.

The resulting distribution of these indices was roughly uniform over a linear scale of 1 to 99; the scale was then divided into 9 intervals symmetrically arranged about 50 and so graduated as to give the nearest approach to a normal frequency distribution.

The most suitable division of the scale was as follows and this was adopted for the correlation tables:

| Proficiency group | I | II | III | IV | V | VI | VII | VIII | IX |
|---|---|---|---|---|---|---|---|---|---|
| Rank in 100 ... ... | 1 | 2—5 | 6—16 | 17—37 | 38—62 | 63—83 | 84—94 | 95—98 | 99 |
| Frequency observed ... | 4 | 43 | 80 | 91 | 101 | 95 | 34 | 18 | 6 |

The distribution is slightly skew; taking the class interval of this scale as unit of proficiency and the centre of group V as origin, the means and standard deviations resulting were as follows:

Category 01. Mean $+ \cdot 3218 \pm \cdot 0860$, $\sigma = 1 \cdot 6806$.
Categories 2 + 3. „ $+ \cdot 4602 \pm \cdot 0675$, $\sigma = 1 \cdot 7005$.
Complete sample. „ $+ \cdot 3415 \pm \cdot 0572$, $\sigma = 1 \cdot 8246$.

The means do not differ significantly. The correlation coefficients between proficiency and thyroid breadth are shown in Table XXVI. There is a correlation with height after age correction, $r = \cdot 2081 \pm \cdot 0300$. After correction for height by partial correlation as in the previous sections, the coefficients with thyroid breadth become

Category 01. $r = \cdot 0413 \pm \cdot 0510$.
Categories 2 + 3. $r = \cdot 0641 \pm \cdot 0395$.
Complete sample. $r = \cdot 0766 \pm \cdot 0312$.

There is therefore no certainly significant correlation with proficiency in school work; the somewhat larger coefficient for the complete sample may be due to the effect of heterogeneity in this sample. The coefficients, however, seem to definitely disprove any association with backwardness.

The regression of proficiency in class on thyroid breadth is shown in Figure 16, the mean of all girls being here taken as the zero line; the actual deviations from this mean, and the original frequencies, are shown in Table XXXII.

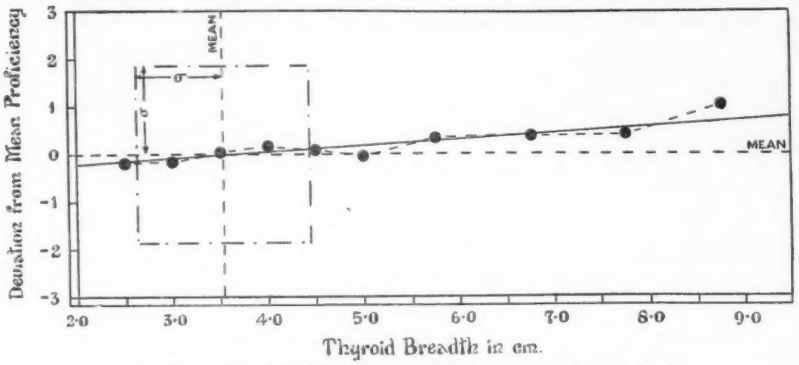



Fig. 16. Regression of proficiency in school work on thyroid breadth in girls: both corrected for age.

TABLE XXXII.

| $B$ cm. = | 2·5— | 3— | 3·5— | 4— | 4·5— | 5— | 5·5— | 6— | 6·5— | 7— | 7·5— | 8— | 8·5— | 9— |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Deviation from mean proficiency | −·191 | −·141 | +·040 | +·177 | +·074 | −·060 | +·536 | +·059 | −·499 | +1·166 | +·221 | +·809 | +1·809 | −1·191 |
| Number of girls | 27 | 59 | 106 | 83 | 66 | 38 | 22 | 28 | 13 | 14 | 17 | 8 | 6 | 2 |

The equation of the regression line is $y = \cdot 1355 (B - 3 \cdot 537)$

## (11) Conclusions.

1. The method of classifying the sizes of thyroid glands by broad categories without measurement is subject to large personal equations and is only to be recommended for a rapid and continuous survey by a single observer. Measurement of circumference or diameter of the neck is also of little value except as a means of registering the progress of a large goitre.

2. Measurement of the maximum breadth of the thyroid gland is recommended as the most reliable and useful index of the actual size of the gland; additional measurements of the vertical heights of the lobes, besides being difficult to carry out, add nothing to accuracy for biometric purposes.

3. Goitre prevalence increases with age to a maximum in boys about 13—14 and in girls about 17—18, and then declines. Where goitre is not endemic the thyroid gland changes very slightly in size from $10\frac{1}{2}$ to $13\frac{1}{2}$, this flattening of the growth curve just before puberty being unique amongst physical measurements so far investigated. In girls a rapid development of the gland occurs between $13\frac{1}{2}$ and 15 which is doubtless associated with the onset of menstruation.

4. After correction for age and the normal variation with height, there remains in *boys* no evidence of any sensible relation between size of the thyroid gland and physical development, size of the head, rate of growth or strength of grip. In *girls*, however, there remains a significant positive association with height and weight, rates of growth, grip, systolic and diastolic pressures, but none with pulse pressure, pulse rate, colour of hair or eyes or proficiency in school work. The regressions of these physical characters on thyroid size for the most part become greater as the size of the goitre increases. The correspondence between growth curves of the thyroid and of diastolic pressure is noteworthy. Long-continued general development in excess of the average rate seems to be favourable to the appearance of large goitres whilst slow followed by temporary rapid growth is associated with small enlargements.

5. The complex relations to rate of physical development in girls brought to light in this and a previous paper (1) are capable of explanation on the theory of iodine deprivation for which they indirectly afford additional proof; the facts cannot be explained by an infective theory unless the infective agent acts by preventing proper absorption of iodine.

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# SUPPLEMENTARY TABLES FOR DETERMINING CORRELATION FROM TETRACHORIC GROUPINGS.

(*First Series.*)

THE following table supplements and replaces the table published in Vol. VIII. pp. 385—395, of this Journal. It supplements it in that this table ranges from $r = \cdot 00$ to $r = 1 \cdot 00$, and it replaces it in that the cell content is now given to six instead of four decimal places. The present table enables the value of the tetrachoric coefficient of correlation to be found for all positive values of $r$ when $h$ and $k$ have the same sign, and for all negative values of $r$ when $h$ and $k$ have different signs. It is hoped that the computer, Dr Alice Lee, may be able to extend her table of 1916* to the values of $r$ between $\cdot 00$ and $-\cdot 80$ for $h$ and $k$ both of the same sign, which will complete the full range of correlation and enable us to determine any tetrachoric coefficient by simple interpolation and without the need in any case of solving a high order equation.

The computation of the present table has been rendered possible by aid from the Royal Society Government Grant Committee.

If we have a bivariate normal system:

$$z = \frac{N}{2\pi\,\sigma_x\sigma_y\sqrt{1-r^2}}\, e^{-\frac{1}{2}\frac{1}{1-r^2}\left(\frac{x^2}{\sigma_x^2}-\frac{2rxy}{\sigma_x\sigma_y}+\frac{y^2}{\sigma_y^2}\right)},$$

and

$$\frac{d}{N} = \frac{1}{2\pi\sqrt{1-r^2}}\int_h^\infty\int_k^\infty e^{-\frac{1}{2}\frac{1}{1-r^2}(x^2-2rxy+y^2)}\,dxdy,$$

then the present table gives the value of $d/N$ in the "quadrant" from $h = x/\sigma_x$, $k = y/\sigma_y$, both up to infinity for values of $r$ proceeding by $\cdot 05$ from $\cdot 00$ to $1 \cdot 00$, and values of $h$ and $k$ from 0 to $2 \cdot 6$ proceeding by $\cdot 1$. The table may be used for two purposes: first to find the tetrachoric coefficient of any fourfold table, and secondly to determine the contents of any cell (or of all the cells) in a bivariate table on the hypothesis that the distribution is normal; this is of course preliminary to testing whether the assumption of normality is legitimate, i.e. to see whether the observed distribution provides a reasonable value in the "goodness of fit" test. For the first purpose undoubtedly a table in which the arguments were $\frac{1}{2}(1-\alpha_1)$, $\frac{1}{2}(1-\alpha_2)$, i.e. the "tail areas" beyond $h$ and $k$, respectively, would be most serviceable, in that these are directly given and we should not have

* *Biometrika*, Vol. XI. pp. 284—291.

to determine $h$ and $k$ to start with for our fourfold table. But when we need to determine cell contents from graduated scales, it seems more reasonable to proceed by $h$ and $k$, which will be known as soon as the means and standard deviations have been found. $h$ and $k$ will then be affected by sampling errors in the means and standard deviations and not by irregularities in the individual marginal cell contents. Of course, theoretical marginal cell contents could be found from these values of $h$ and $k$, and thus finally a table of $\frac{1}{2}(1-\alpha_1)$ and $\frac{1}{2}(1-\alpha_2)$ used, but the labour on the whole would be, we think, greater, although it would certainly be less in finding a tetrachoric correlation coefficient from the latter table.

First: To determine a tetrachoric coefficient of correlation we have primarily to determine the $d/N$, the $\frac{1}{2}(1-\alpha_1)$, and the $\frac{1}{2}(1-\alpha_2)$; from the two latter we determine $h$ and $k$. Hence from a knowledge of $d/N$, $h$ and $k$ we have to find $r$. Now this clearly is a case of determining $r$ from a table of triple argument, and thus the problem reduces to the choice of an appropriate method of interpolation. That method depends on the number of decimal places required in the answer. We may note the following points:

(*a*) Linear interpolation with regard to the argument $r$ appears for the whole range of examples on which we have tested the table to be adequate.

(*b*) Hyperbolic interpolation* with regard to the arguments $h$ and $k$ is practically adequate for obtaining the value of $r$ to three decimal places throughout the table. It is not adequate for obtaining $r$ correctly to four decimal places.

(*c*) To obtain $r$ correct to four decimal places we must use a central difference formula as far as the $\delta^3$'s. A forward difference formula going as far as the first $\Delta$'s was found to give poorer results than a hyperbolic interpolation.

(*d*) At the boundaries of the table, where a result to four-figure accuracy is required, it is needful to use a central difference finial or boundary formula.

It would undoubtedly have been a gain, if a single formula, i.e. the symmetrical hyperbolic interpolation, had sufficed, but this would have necessitated a table with the $h$ and $k$ intervals reduced to ·05, that is to say, a table occupying 64 and not 32 pages and so beyond our powers of publishing.

The following are the interpolation formulae needful for an effective use of the table. The diagrams explain the notation. Differences involving the first subscript are written $\delta^2$, and those involving the second subscript $\delta'^2$. As it was impossible on account of cost to publish the differences the reader must bear in mind that:

$$\delta^2 z_{r,s} = z_{r+1,s} + z_{r-1,s} - 2z_{r,s},$$
$$\delta'^2 z_{r,s} = z_{r,s+1} + z_{r,s-1} - 2z_{r,s},$$

and these are easily obtained from the table by aid of any calculating machine.

* The hyperbolic interpolation corresponds to fitting the hyperbolic paraboloid $z = a_0 + a_1 x + a_2 y + a_3 xy$ to the four nearest entries.

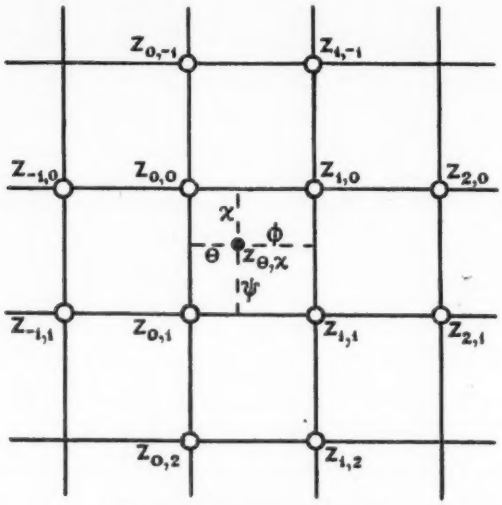
**Formulae.**

($\alpha$) *Hyperbolic Interpolating Formula.*

$$z_{\theta,\chi} = \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}.$$

($\beta$) *Midpanel Central Difference Formula.* (Correct to third differences.)

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & - \tfrac{1}{6}\psi\chi\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}. \end{aligned}$$



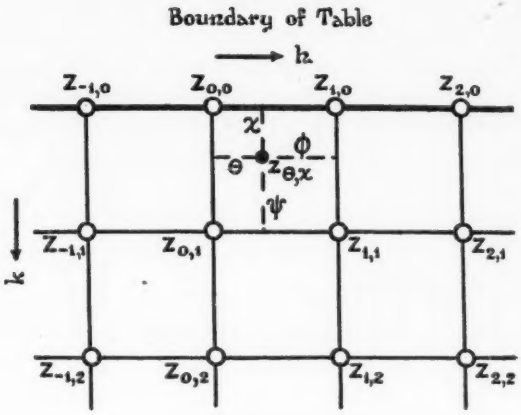

**FORMULAE (α) & (β)**
**ORDINARY NON-FINIAL PANEL**
**(NEITHER h NOR k LIMITED)**

Fig. 1.

($\gamma$) *Midpanel Central Difference Singly Finial Formula**. (Correct to third differences.)

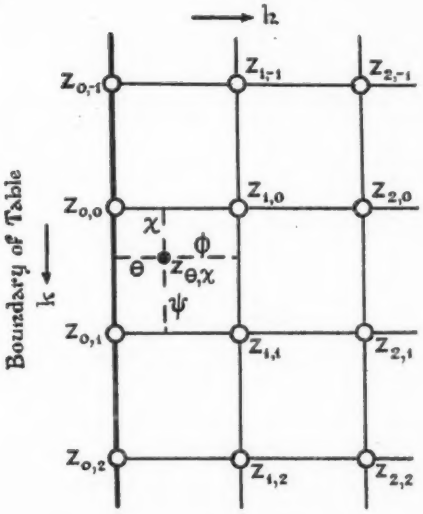
$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\{(1+\phi)(\psi\delta^2 z_{0,0} + \chi\delta^2 z_{0,1}) + (1+\theta)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\} \\ & - \tfrac{1}{6}\psi\chi\{(4+\psi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1}) - (1+\psi)(\phi\delta'^2 z_{0,2} + \theta\delta'^2 z_{1,2})\}. \end{aligned}$$

* This formula applies when our value of $h$ (corresponding to $\delta'^2$ and the second subscript) lies between ·00 and ·10.




FORMULA (γ)
SINGLY FINIAL (k LIMITED)

Fig. 2.




FORMULA (γ) *bis*
SINGLY FINIAL (h LIMITED)

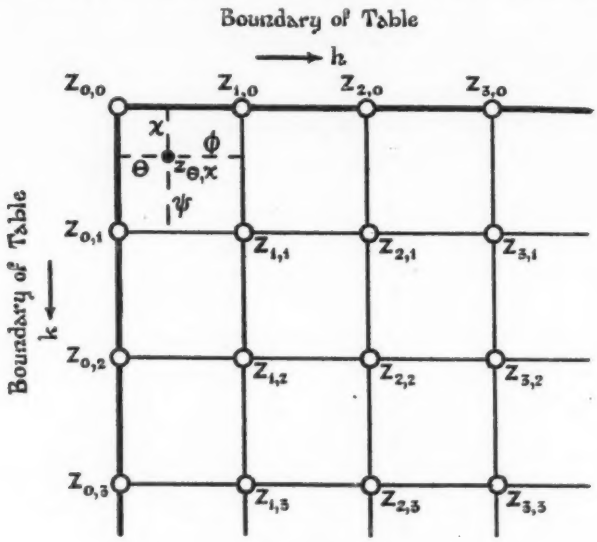
Fig. 3.

($\gamma$)$^{bis}$ *Midpanel Central Difference Singly Finial Formula**. (Correct to third differences.)

$$z_{\theta,x} = \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}$$
$$- \tfrac{1}{6}\theta\phi\,[(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})]$$
$$- \tfrac{1}{6}\psi\chi\,[(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})].$$

($\delta$) *Midpoint Central Difference Doubly Finial Formula†*. (Correct to third differences.)

$$z_{\theta,x} = \phi\psi z_{0,0} + \phi\chi z_{0,1} + \theta\psi z_{1,0} + \theta\chi z_{1,1}$$
$$- \tfrac{1}{6}\theta\phi\,[(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})]$$
$$- \tfrac{1}{6}\psi\chi\,[(4+\psi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1}) - (1+\psi)(\phi\delta'^2 z_{0,2} + \theta\delta'^2 z_{1,2})].$$




FORMULA (δ)

DOUBLY FINIAL PANEL (h AND k LIMITED)

Fig. 4.

In all cases a knowledge of $h$ and $k$ to five decimal places will be adequate. This can be found by linear interpolation from the table in *Biometrika*, Vol. VII. pp. 442—451, or in the book, *Tables for Statisticians*, Table XXIX, pp. 42—51, where the value of $h$ to five decimal places is given for each value of $\frac{1}{2}(1-\alpha)$ from ·001 to ·500 by increments of ·001. We shall refer to this as Table XXIX.

* This formula applies when our value of $h$ (corresponding to $\delta^2$ and the first subscript) lies between ·00 and ·10.

† This formula applies when both $h$ and $k$ lie between ·00 and ·10.

It will be seen by our diagrams that $\theta$ and $\phi$ refer to variation of $h$, and $\chi$ and $\psi$ to variation of $k$. Further the terms of ($\alpha$) are common to all the formulae, and we propose to consider the degree of accuracy with which they suffice to give the correct value of $r$ at different parts of the table.

*Illustration I. Intelligence and Enlarged Glands. (Boys.)*

Intelligence.

| State of Glands. | Intelligent and Over | Slow Intelligent and Under | Totals |
|---|---|---|---|
| Normal | 241 | 215 | 456 |
| Enlarged | 82 | 81 | 163 |
| Totals | 323 | 296 | 619 |

$\frac{1}{2}(1-\alpha_h) = \frac{296}{619} = \cdot 478{,}191, \quad \frac{1}{2}(1-\alpha_k) = \frac{163}{619} = \cdot 263{,}328.$

Hence by Table XXIX, using linear interpolation,

$$h = \cdot 05469, \quad k = \cdot 63312.$$

Further: $$d/N = \frac{81}{619} = \cdot 130{,}8562.$$

From these arguments we are to find $r$. Clearly $h$ lies between $\cdot 0$ and $\cdot 1$ and $k$ between $\cdot 6$ and $\cdot 7$. The value of $d$ for these values of $h$ and $k$ might lie in the tables for $r = \cdot 00$, $r = \cdot 05$ or $r = \cdot 10$. The latter is very improbable because the point corresponding to our $h$ and $k$ lies in the triangle of which the $z$'s are $\cdot 150\ldots$, $\cdot 139\ldots$ and $\cdot 133\ldots$, all greater than our $d/N = \cdot 130\ldots$. It is safest, however, to start interpolating from the mid-table $r = \cdot 05$.

We have $\theta = \cdot 5469$, $\phi = \cdot 4531$, $\chi = \cdot 3312$, $\psi = \cdot 6688$ and the scheme:

| | $h = 0 \cdot 0$ | $h = -0 \cdot 1$ |
|---|---|---|
| $k = 0 \cdot 6$ | $\cdot 143{,}775\ (z_{0,0})$ | $\cdot 132{,}829\ (z_{1,0})$ |
| $k = 0 \cdot 7$ | $\cdot 127{,}212\ (z_{0,1})$ | $\cdot 117{,}555\ (z_{1,1})$ |

We then arrange our work thus, using formula ($\alpha$):

$$\left.\begin{matrix} d/N = \cdot 6688 \times \cdot 4531 \\ \cdot 3030{,}3228 \end{matrix}\right\} \left\{\begin{matrix} \cdot 143{,}775 + \cdot 6688 \times \cdot 5469 \\ \cdot 137{,}127 \quad \cdot 3657{,}6672 \end{matrix}\right\} \left\{\begin{matrix} \cdot 132{,}829 \\ \cdot 126{,}204 \end{matrix}\right.$$

$$\left.\begin{matrix} + \cdot 3312 \times \cdot 4531 \\ \cdot 1500{,}6672 \end{matrix}\right\} \left\{\begin{matrix} \cdot 127{,}212 + \cdot 3312 \times \cdot 5469 \\ \cdot 120{,}984 \quad \cdot 1811{,}3328 \end{matrix}\right\} \left\{\begin{matrix} \cdot 117{,}555 \\ \cdot 111{,}345 \end{matrix}\right.$$

The lower number in the curled brackets facing left is the product of the $\theta$, $\phi$, $\chi$, $\psi$ pairs standing above them. Taking the upper numbers in the brackets facing right and multiplying them continuously on the machine we find

$$d/N = \cdot 132{,}5363.$$

This value is too high; we must therefore reduce the correlation by taking the table for $r = \cdot 00$. The corresponding values for the $z$'s are those in the lower line of the curled brackets facing right, and repeating the continuous multiplication process

on the machine we have $d/N = \cdot 126{,}0388$. Interpolating linearly for $d/N = \cdot 130{,}8562$ we find

$$r = \cdot 00 + \cdot 05 \times \frac{48174}{64975} = \cdot 0371.$$

This result would be close enough for the great majority of practical investigations, but the true value of $r$ is $\cdot 0379$. It is of interest to determine how the table result may be brought into accord with the actual value of $r$. Our $d/N$ is in a finial panel of the table; we cannot get the central difference for $z_{0,0}$ and $z_{0,1}$ as we are on the border of the table $h = 0 \cdot 0$. Accordingly, let us try the forward difference formula:

($\epsilon$) *Forward Difference Formula.* (Correct to second differences.)

$$\begin{aligned} z_{\theta,\chi} = z_{0,0} &+ \theta\,(z_{1,0} - z_{0,0}) + \chi\,(z_{0,1} - z_{0,0}) \\ &+ \tfrac{1}{2}\,[\theta\,(\theta - 1)\,(z_{2,0} - 2z_{1,0} + z_{0,0}) + 2\theta\chi\,(z_{1,1} - z_{0,1} - z_{1,0} + z_{0,0}) \\ &+ \chi\,(\chi - 1)\,(z_{0,2} - 2z_{0,1} + z_{0,0})]. \end{aligned}$$

Here $\theta = \cdot 5469$ and $\chi = \cdot 3312$ as before, and we have for the two tables $r = \cdot 00$ and $r = \cdot 05$:

$$\begin{aligned} z_{\theta,\chi} = &\begin{Bmatrix} \cdot 137{,}127 \\ \cdot 143{,}775 \end{Bmatrix} + \cdot 5469 \begin{Bmatrix} - \cdot 010{,}923 \\ - \cdot 010{,}946 \end{Bmatrix} + \cdot 3312 \begin{Bmatrix} - \cdot 016{,}145 \\ - \cdot 016{,}563 \end{Bmatrix} \\ &- \cdot 1239{,}0019 \begin{Bmatrix} \cdot 000{,}108 \\ \cdot 000{,}043 \end{Bmatrix} - \cdot 1107{,}5328 \begin{Bmatrix} \cdot 001{,}091 \\ \cdot 001{,}058 \end{Bmatrix} \\ &+ \cdot 1811{,}3228 \begin{Bmatrix} \cdot 001{,}286 \\ \cdot 001{,}289 \end{Bmatrix} \\ = &\cdot 125{,}905, \text{ from upper line,} \\ = &\cdot 132{,}414, \text{ from lower line.} \end{aligned}$$

Interpolating linearly, $r = \cdot 00 + \cdot 05\,\frac{4951}{6509} = \cdot 0380$, which is in good agreement with the value $\cdot 0379$ found by developing the tetrachoric equation and solving it as in the old method.

Let us now use the appropriate central difference finial formula $(\gamma)^{bis}$. We require the values of the following eight $\delta^2$'s and $\delta'^2$'s for the two tables of $r$*:

| | $r = \cdot 00$ | $r = \cdot 05$ |
|---|---|---|
| $\delta^2 z_{1,0}$ | $\cdot 000{,}108$ | $\cdot 000{,}043$ |
| $\delta^2 z_{1,1}$ | $\cdot 000{,}096$ | $\cdot 000{,}034$ |
| $\delta^2 z_{2,0}$ | $\cdot 000{,}215$ | $\cdot 000{,}151$ |
| $\delta^2 z_{2,1}$ | $\cdot 000{,}189$ | $\cdot 000{,}129$ |
| $\delta'^2 z_{0,0}$ | $\cdot 000{,}997$ | $\cdot 000{,}956$ |
| $\delta'^2 z_{1,0}$ | $\cdot 000{,}917$ | $\cdot 000{,}876$ |
| $\delta'^2 z_{0,1}$ | $\cdot 001{,}091$ | $\cdot 001{,}058$ |
| $\delta'^2 z_{1,1}$ | $\cdot 001{,}004$ | $\cdot 000{,}971$ |

* It is extremely easy to obtain by the machine the central differences of any table entry. To get $\delta^2$ for any entry, add the entries to right and left of the given entry, place the entry itself on the machine and subtract it twice; the result on the slide is the $\delta^2$. To get $\delta'^2$ for any entry, add the entries above and below it and subtract twice the given entry. The discovery of either $\delta^2$ or $\delta'^2$ is thus a single continuous operation, and one of great simplicity and rapidity.

We need only compute the second and third lines of $(\gamma)^{bis}$, as we have already obtained the values of the first. Calling this remainder $R$, we have:

$$R = -\cdot0413{,}0006\left[4\cdot4531\left(\cdot6688\begin{Bmatrix}\cdot000{,}108\\ \cdot000{,}043\end{Bmatrix}+\cdot3312\begin{Bmatrix}\cdot000{,}096\\ \cdot000{,}034\end{Bmatrix}\right)\right.$$
$$\left.-1\cdot5469\left(\cdot6688\begin{Bmatrix}\cdot000{,}215\\ \cdot000{,}151\end{Bmatrix}+\cdot3312\begin{Bmatrix}\cdot000{,}189\\ \cdot000{,}129\end{Bmatrix}\right)\right]$$
$$-\cdot0369{,}1776\left[1\cdot6688\left(\cdot4531\begin{Bmatrix}\cdot000{,}997\\ \cdot000{,}956\end{Bmatrix}+\cdot5469\begin{Bmatrix}\cdot000{,}917\\ \cdot000{,}876\end{Bmatrix}\right)\right.$$
$$\left.+1\cdot3312\left(\cdot4531\begin{Bmatrix}\cdot001{,}091\\ \cdot001{,}058\end{Bmatrix}+\cdot5469\begin{Bmatrix}\cdot001{,}004\\ \cdot000{,}971\end{Bmatrix}\right)\right],$$

the upper line in the curled brackets referring to $r = \cdot00$, and the lower to $r = \cdot05$. These give:

$$R = \begin{cases}-\cdot000{,}1159\\ -\cdot000{,}1040\end{cases},$$

which must be taken away from the hyperbolic formula values of $d/N$, i.e. ·126,0388 and ·132,5363 respectively.

Thus we get

$$d/N = \cdot125{,}9229 \quad \text{for} \quad r = 0\cdot00,$$

and

$$\cdot132{,}4323 \quad \text{for} \quad r = 0\cdot05.$$

Hence, interpolating for $d/N = \cdot130{,}8562$ we find $r = \cdot0379$, the correct value. We accordingly conclude that the correct value in a finial panel will be obtained by the central difference formula $(\gamma)^{bis}$, although it fails to be given by a forward difference formula.

*Illustration II. Intelligence and General Nutrition.* (*Girls.*)

Intelligence.

| Nutrition. | Intelligent and Above | Slow Intelligent and Under | Totals |
|---|---|---|---|
| Good ... ... | 91 | 171 | 262 |
| Medium to Bad | 102 | 221 | 323 |
| Totals | 193 | 392 | 585 |

In this case $\frac{1}{2}(1-\alpha_h) = \cdot329{,}914$, $\frac{1}{2}(1-\alpha_k) = \cdot447{,}863$,

leading by aid of Table XXIX to

$$h = \cdot44015, \quad k = \cdot13107,$$

while

$$d/N = \frac{91}{585} = \cdot155{,}5556.$$

Clearly

$$\theta = \cdot4015, \quad \phi = \cdot5985, \quad \chi = \cdot3107, \quad \psi = \cdot6893.$$

Between the values $h = \cdot4$ and $\cdot5$ and $k = \cdot1$ and $\cdot2$, our value for $d/N$ occurs in the $r$ tables for values ·00, ·05, ·10 and ·15 but the $h$ and $k$ values are nearer to ·4 and ·1 than ·5 and ·2. This excludes $r = \cdot15$, as clearly $d/N = \cdot155\ldots$ lies

more than half-way from ·4 to ·5 and ·1 to ·2, i.e. the diagonal right top to left bottom corner is from ·163... to ·166.... Accordingly we start with interpolating into the $r = ·05$ table. We have

$$d/N = z_{\theta,\chi} = \begin{matrix} ·6893 \times ·5985 \\ ·4125,4605 \end{matrix} \left\{ \begin{matrix} ·165,835 + ·6893 \times ·4015 \\ ·173,234 \quad ·2767,5395 \end{matrix} \right\} \left\{ \begin{matrix} ·148,979 \\ ·156,008 \end{matrix} \right.$$
$$+ \begin{matrix} ·3107 \times ·5985 \\ ·1859,5395 \end{matrix} \left\{ \begin{matrix} ·152,195 + ·3107 \times ·4015 \\ ·159,456 \quad ·1247,4605 \end{matrix} \right\} \left\{ \begin{matrix} ·136,717 \\ ·143,667 \end{matrix} \right.$$

The upper numbers in the right-facing curled brackets give $d/N = ·155,0218$. This is not large enough, and we accordingly choose the $r = ·10$ table for our second interpolation and this gives $d/N = ·162,2162$. Finally, interpolating linearly for $d/N = ·155,5556$, we have

$$r = ·05 + ·05 \times \frac{5338}{71944} = ·0537.$$

The true value of $r$ is ·0542.

If we proceed by the forward difference formula, we find $r = ·0545$, again a result slightly in excess of the true value. As our value of $d/N$ does not fall into a finial panel, formula ($\beta$) applies, and we proceed to calculate the additional terms. We find:

| | $r = ·05$ | $r = ·10$ |
|---|---|---|
| $\delta^2 z_{0,0}$ | ·000,614 | ·000,552 |
| $\delta^2 z_{0,1}$ | ·000,756 | ·000,703 |
| $\delta^2 z_{1,0}$ | ·000,558 | ·000,496 |
| $\delta^2 z_{1,1}$ | ·000,687 | ·000,634 |
| $\delta'^2 z_{0,0}$ | ·000,063 | − ·000,010 |
| $\delta'^2 z_{1,0}$ | ·000,053 | − ·000,017 |
| $\delta'^2 z_{0,1}$ | ·000,201 | ·000,130 |
| $\delta'^2 z_{1,1}$ | ·000,175 | ·000,107 |

Substituting in the formula in a manner suitable for a nearly continuous operation we obtain—the upper figures in curled brackets referring to $r = ·05$, and the lower to $r = ·10$ interpolation:

$$R = \begin{matrix} -\frac{1}{6} ·4015 \times ·5985 \\ - ·0400,4963 \end{matrix} \Bigg[ 1·5985 \left( ·6893 \left\{ \begin{matrix} ·000,614 \\ ·000,552 \end{matrix} \right\} + ·3107 \left\{ \begin{matrix} ·000,558 \\ ·000,496 \end{matrix} \right\} \right)$$
$$+ 1·4015 \left( ·6893 \left\{ \begin{matrix} ·000,756 \\ ·000,703 \end{matrix} \right\} + ·3107 \left\{ \begin{matrix} ·000,687 \\ ·000,634 \end{matrix} \right\} \right) \Bigg]$$
$$\begin{matrix} -\frac{1}{6} ·3107 \times ·6893 \\ - ·0356,9425 \end{matrix} \Bigg[ 1·6893 \left( ·5985 \left\{ \begin{matrix} ·000,063 \\ - ·000,010 \end{matrix} \right\} + ·4015 \left\{ \begin{matrix} ·000,053 \\ - ·000,017 \end{matrix} \right\} \right)$$
$$+ 1·3107 \left( ·5985 \left\{ \begin{matrix} ·000,201 \\ ·000,130 \end{matrix} \right\} + ·4015 \left\{ \begin{matrix} ·000,175 \\ ·000,107 \end{matrix} \right\} \right) \Bigg]$$
$$= - ·0400,4963 \left[ \left\{ \begin{matrix} ·000,9537 \\ ·000,8546 \end{matrix} \right\} + \left\{ \begin{matrix} ·001,0295 \\ ·000,9552 \end{matrix} \right\} \right]$$
$$- ·0356,9425 \left[ \left\{ \begin{matrix} ·000,0996 \\ - ·000,0216 \end{matrix} \right\} + \left\{ \begin{matrix} ·000,2498 \\ ·000,1583 \end{matrix} \right\} \right]$$
$$= - \left\{ \begin{matrix} ·000,0794 \\ ·000,0072 \end{matrix} \right\} - \left\{ \begin{matrix} ·000,0125 \\ ·000,0049 \end{matrix} \right\} = - \left\{ \begin{matrix} ·000,0919 \\ ·000,0121 \end{matrix} \right\}.$$

Accordingly:

$$\text{For } r = \cdot 05: \quad d/N = \cdot 155{,}0218 - \cdot 000{,}0919 = \cdot 154{,}9299,$$
$$\text{„ } \quad r = \cdot 10: \quad d/N = \cdot 162{,}2162 - \cdot 000{,}0121 = \cdot 162{,}2041,$$

and finally for $d/N = \cdot 155{,}5556$,

$$r = \cdot 05 + \cdot 05 \times \frac{6257}{72742} = \cdot 0543.$$

The result is thus in excellent agreement with the result found by forming the high order tetrachoric equation and solving it. We conclude that formula ($\alpha$) will suffice for three and formula ($\beta$) or formula ($\gamma$) for four-figure accuracy. We shall give further illustrations with greater brevity as the method of arranging the work will now be clear to the reader.

*Illustration III. Mother's Habits and Baby's Health.*

Mother's Habits.

| Baby's Health. | Good | Not Good | Totals |
|---|---|---|---|
| Good ... | 625 | 218 | 843 |
| Not Good | 206 | 136 | 342 |
| Totals | 831 | 354 | 1185 |

Here $\frac{1}{2}(1-\alpha_h) = \cdot 298{\cdot}734, \quad \frac{1}{2}(1-\alpha_k) = \cdot 288{,}608,$

and thus $h = \cdot 52805, \quad k = \cdot 55746,$

while $d/N = \cdot 114{,}7679,$

$$\theta = \cdot 2805, \quad \phi = \cdot 7195, \quad \chi = \cdot 5746, \quad \psi = \cdot 4254.$$

Between $h = \cdot 50$ and $\cdot 60$, and $k = \cdot 50$ and $\cdot 60$, the above value of $d/N$ occurs in tables of $r$ for $\cdot 20$, $\cdot 25$ and $\cdot 30$, but not in those for $r = \cdot 15$ and $\cdot 35$. We therefore start by interpolating into $\cdot 25$. Using formula ($\alpha$)

$$d/N = \cdot 3060{,}7530 \begin{Bmatrix} \cdot 127{,}375 \\ \cdot 120{,}715 \end{Bmatrix} + \cdot 4134{,}2470 \begin{Bmatrix} .115{,}238 \\ \cdot 108{,}878 \end{Bmatrix}$$
$$+ \cdot 1193{,}2470 \begin{Bmatrix} \cdot 115{,}238 \\ \cdot 108{,}878 \end{Bmatrix} + \cdot 1611{,}7530 \begin{Bmatrix} \cdot 104{,}390 \\ \cdot 098{,}392 \end{Bmatrix}.$$

The first or upper series of numbers gives us $d/N = \cdot 117{,}2044$, or too high a value. Hence the second system is inserted for $r = \cdot 20$. This gives $d/N = \cdot 110{,}7964$. Thus, by linear interpolation,

$$r = \cdot 20 + \cdot 05 \times \frac{39715}{64080} = \cdot 2310.$$

The true value is $\cdot 2317$.

We now examine this result by aid of formula ($\beta$). The $\delta^2$'s and $\delta'^2$'s are as follows:

| | $r = \cdot 20$ | $r = \cdot 25$ |
|---|---|---|
| $\delta^2 z_{0,0} = \delta'^2 z_{0,0}$ | $\cdot 000{,}337$ | $\cdot 000{,}278$ |
| $\delta^2 z_{1,0} = \delta'^2 z_{0,1}$ | $\cdot 000{,}284$ | $\cdot 000{,}395$ |
| $\delta^2 z_{0,1} = \delta'^2 z_{1,0}$ | $\cdot 000{,}445$ | $\cdot 000{,}227$ |
| $\delta^2 z_{1,1} = \delta'^2 z_{1,1}$ | $\cdot 000{,}383$ | $\cdot 000{,}333$ |

Hence

$$R = -\cdot0336,3662_{(5)}\left[1\cdot7195\left(\cdot4254\begin{Bmatrix}\cdot000,337\\ \cdot000,278\end{Bmatrix}+\cdot5746\begin{Bmatrix}\cdot000,445\\ \cdot000,227\end{Bmatrix}\right)\right.$$
$$\left.+1\cdot2805\left(\cdot4254\begin{Bmatrix}\cdot000,284\\ \cdot000,395\end{Bmatrix}+\cdot5746\begin{Bmatrix}\cdot000,383\\ \cdot000,333\end{Bmatrix}\right)\right]$$
$$-\cdot0407,3914\left[1\cdot4254\left(\cdot7195\begin{Bmatrix}\cdot000,337\\ \cdot000,278\end{Bmatrix}+\cdot2805\begin{Bmatrix}\cdot000,445\\ \cdot000,227\end{Bmatrix}\right)\right.$$
$$\left.+1\cdot5746\left(\cdot7195\begin{Bmatrix}\cdot000,284\\ \cdot000,395\end{Bmatrix}+\cdot2805\begin{Bmatrix}\cdot000,383\\ \cdot000,333\end{Bmatrix}\right)\right]$$
$$=-\cdot000,0791 \text{ for } r=\cdot20, \text{ and } =-\cdot000,0694 \text{ for } r=\cdot25.$$

Accordingly we have

$$d/N=\cdot110,7173 \text{ for } r=\cdot20 \text{ and } =\cdot117,1350 \text{ for } r=\cdot25.$$

Interpolating linearly for $d/N=\cdot114,7679$, we find

$$r=\cdot20+\cdot05\times\frac{40506}{64177}=\cdot2316.$$

The result from the tetrachoric equation is ·2317.

If we use the forward difference formula ($\epsilon$) we find $r=\cdot2316$, also a quite good result.

*Illustration IV. Ventilation of Home and Health of Mother.*

Health of Mother.

| Ventilation. | Good | Indifferent and Bad | Totals |
|---|---|---|---|
| Good and Fair | 1080 | 352 | 1432 |
| Poor ... ... | 133 | 131 | 264 |
| Totals | 1213 | 483 | 1696 |

Here $\frac{1}{2}(1-\alpha_h)=\cdot284,788$, $\frac{1}{2}(1-\alpha_k)=\cdot155,660$,

while $d/N=\cdot077,2406$.

We find $h=\cdot56868$, $k=1\cdot01245$,

$\theta=\cdot6868$, $\phi=\cdot3132$, $\chi=\cdot1245$, $\psi=\cdot8755$.

The required value of $d/N$ will be found in the section $h=\cdot5$ to $\cdot6$ and $k=1\cdot0$ to $1\cdot1$ in the tables for $r=\cdot35$, $\cdot40$ and $\cdot45$. We therefore first determine $d/N$ for $\cdot40$.

$z_{\theta,\chi}$ from ($\alpha$) is given by

$$z_{\theta,\chi}=\cdot2742,0660\begin{Bmatrix}\cdot086,679\\ \cdot081,519\end{Bmatrix}+\cdot6012,9340\begin{Bmatrix}.079,913\\ \cdot074,857\end{Bmatrix}$$
$$+\cdot0389,9340\begin{Bmatrix}.075,987\\ \cdot071,320\end{Bmatrix}+\cdot0855,0660\begin{Bmatrix}\cdot070,212\\ \cdot065,619\end{Bmatrix}.$$

$d/N=\cdot080,7857$ for $r=\cdot40$, or upper figures. This is too high; we therefore take the value for $\cdot35$, the lower figures. These give

$$d/N=\cdot075,7559.$$

Hence by linear interpolation

$$r = \cdot35 + \cdot05 \times \frac{14847}{50298} = \cdot3648.$$

The exact value is $r = \cdot3653$.

The answer is correct to three figures. To get a better approximation we must proceed to second differences.

Let us try forward differences in this case using formula ($\epsilon$). We have

$$\begin{aligned} z_{\theta,\chi} = & \begin{Bmatrix} \cdot081,519 \\ \cdot086,670 \end{Bmatrix} + \cdot6868 \begin{Bmatrix} -\cdot006,662 \\ -\cdot006,757 \end{Bmatrix} + \cdot1245 \begin{Bmatrix} -\cdot010,199 \\ -\cdot010,683 \end{Bmatrix} \\ & - \cdot1075,5288 \begin{Bmatrix} \cdot000,051 \\ -\cdot000,013 \end{Bmatrix} + \cdot0855,0660 \begin{Bmatrix} \cdot000,961 \\ \cdot000,982 \end{Bmatrix} \\ & - \cdot0544,9988 \begin{Bmatrix} \cdot000,758 \\ \cdot000,747 \end{Bmatrix} \\ = & \cdot075,709, \text{ for } r = \cdot35, \\ = & \cdot086,670, \text{ for } r = \cdot40. \end{aligned}$$

Hence, by linear interpolation for $d/N = \cdot077,241$, we have

$$r = \cdot35 + \cdot05 \times \frac{1532}{5035} = \cdot3652.$$

The result could scarcely agree better with the actual value $r = \cdot3653$ found from the tetrachoric equation had we proceeded by the central difference instead of a forward difference formula. For a table of this type the latter formula gives very fair results and can be applied to any panel whatever*.

*Illustration V. Cleanliness of Home and Mother's Health.*

| Cleanliness of Home. | Health of Mother. Good | Indifferent and Bad | Totals |
|---|---|---|---|
| Clean ... ... | 939 | 234 | 1173 |
| "Fair" and Dirty | 274 | 249 | 523 |
| Totals | 1213 | 483 | 1696 |

Here $\frac{1}{2}(1-\alpha_h) = \cdot284,788$, $\frac{1}{2}(1-\alpha_k) = \cdot308,373$,

$d/N = \cdot146,8160$,

Hence $h = \cdot56868$, $k = \cdot50047$.

We have $\theta = \cdot6868$, $\phi = \cdot3132$, $\chi = \cdot0047$, $\psi = \cdot9953$.

* It is clearly less exact than the central difference formula in that it neglects third differences, but these do not seem to reach practical importance in any of the cases tested in this introduction.

The value of $d/N$ occurs in the tables for $h$ and $k$ between ·5 and ·6 for $r = ·40$, ·45, ·50 and ·55. The last clearly need not be considered as $k$ is only slightly over ·5 and a value such as ·146... could not occur in this region. We may also discard the first ($r = ·40$), for in the close neighbourhood of $k = ·50$ the value $d/N = ·146...$ could only occur when $h$ is slightly greater than ·50. Working with the ·45 and ·50 table we have

$$z_{\theta,\chi} = ·3117,2796 \begin{Bmatrix} ·155,684 \\ ·162,320 \end{Bmatrix} + ·6835,7204 \begin{Bmatrix} ·142,361 \\ ·149,694 \end{Bmatrix}$$
$$+ ·0014,7204 \begin{Bmatrix} ·142,361 \\ ·149,694 \end{Bmatrix} + ·0032,2796 \begin{Bmatrix} ·130,483 \\ ·137,570 \end{Bmatrix}$$
$$= \begin{Bmatrix} ·146,4758 \\ ·153,9025 \end{Bmatrix}.$$

Interpolating linearly we have

$$r = ·45 + ·05 \times \frac{3402}{74267} = ·4523.$$

The value found from the full tetrachoric equation is ·4524.

Or, the agreement is complete without passing to second order differences.

*Illustration VI. Eye Colour in Brother and Sister.*

Brother.

| Sister. | | Light Eyes | Dark Eyes | Totals |
|---|---|---|---|---|
| | Light Eyes | 616 | 219 | 835 |
| | Dark Eyes | 293 | 372 | 665 |
| | Totals | 909 | 591 | 1500 |

Here $\frac{1}{2}(1 - \alpha_h) = ·394,000$, $\frac{1}{2}(1 - \alpha_k) = ·443,333$,

$$d/N = ·248,0000,$$

Hence $h = ·26891$, $k = ·14253$.

This value of $d/N$ can lie in the ·40, ·45, or ·50 tables for $r$, we therefore start by interpolating into that for $r = ·45$. We have

$$\theta = ·6891, \quad \phi = ·3109, \quad \dot{\chi} = ·4253, \quad \psi = ·5747.$$

Thus $$z_{\theta,\chi} = ·1786,7423 \begin{Bmatrix} ·266,295 \\ ·275,161 \end{Bmatrix} + ·3960,2577 \begin{Bmatrix} ·246,755 \\ ·255,392 \end{Bmatrix}$$
$$+ ·1322,2577 \begin{Bmatrix} ·248,905 \\ ·257,709 \end{Bmatrix} + ·2930,7423 \begin{Bmatrix} ·231,089 \\ ·239,718 \end{Bmatrix}.$$

Hence $$z_{\theta,\chi} = \begin{Bmatrix} ·245,9366 \\ ·254,6369 \end{Bmatrix},$$

the upper number showing that we need $r = ·50$, and not ·40.

Thus by linear interpolation

$$r = \cdot 45 + \cdot 05 \times \frac{20634}{87003} = \cdot 4619.$$

The actual value of $r$ is ·4614.

We will next inquire how much closer we get to this result by using a higher difference formula. The following $\delta^2$'s and $\delta'^2$'s are needed for $(\beta)$:

| | $r = \cdot 45$ | $r = \cdot 50$ |
|---|---|---|
| $\delta^2 z_{0,0}$ | − ·000,407 | − ·000,508 |
| $\delta^2 z_{1,0}$ | − ·000,176 | − ·000,278 |
| $\delta^2 z_{0,1}$ | − ·000,426 | − ·000,539 |
| $\delta^2 z_{1,1}$ | − ·000,225 | − ·000,330 |
| $\delta'^2 z_{0,0}$ | − ·000,614 | − ·000,727 |
| $\delta'^2 z_{1,0}$ | − ·000,611 | − ·000,722 |
| $\delta'^2 z_{0,1}$ | − ·000,426 | − ·000,539 |
| $\delta'^2 z_{1,1}$ | − ·000,444 | − ·000,556 |

Hence

$$R = -\cdot 0357{,}0686_{(5)} \left[ 1\cdot 3109 \left( \cdot 5747 \begin{Bmatrix} - \cdot 000{,}407 \\ - \cdot 000{,}508 \end{Bmatrix} + \cdot 4253 \begin{Bmatrix} - \cdot 000{,}426 \\ - \cdot 000{,}539 \end{Bmatrix} \right) \right.$$
$$\left. + 1\cdot 6891 \left( \cdot 5747 \begin{Bmatrix} - \cdot 000{,}176 \\ - \cdot 000{,}278 \end{Bmatrix} + \cdot 4253 \begin{Bmatrix} - \cdot 000{,}225 \\ - \cdot 000{,}330 \end{Bmatrix} \right) \right]$$
$$- \cdot 0407{,}3665 \left[ 1\cdot 5747 \left( \cdot 3109 \begin{Bmatrix} - \cdot 000{,}614 \\ - \cdot 000{,}727 \end{Bmatrix} + \cdot 6891 \begin{Bmatrix} - \cdot 000{,}611 \\ - \cdot 000{,}722 \end{Bmatrix} \right) \right.$$
$$\left. + 1\cdot 4253 \left( \cdot 3109 \begin{Bmatrix} - \cdot 000{,}426 \\ - \cdot 000{,}539 \end{Bmatrix} + \cdot 6891 \begin{Bmatrix} - \cdot 000{,}444 \\ - \cdot 000{,}556 \end{Bmatrix} \right) \right].$$

Thus $R = \cdot 000{,}0960$, for $r = \cdot 45$,
$= \cdot 000{,}1209$, for $r = \cdot 50$.

These give $d/N = \cdot 246{,}0326$, for $r = \cdot 45$,
$d/N = \cdot 254{,}7578$, for $r = \cdot 50$.

Whence, interpolating for $d/N = \cdot 248{,}0000$, we have

$$r = \cdot 45 + \cdot 05 \times \frac{19674}{87252} = \cdot 4613,$$

only differing by a unit in the last figure from the actual value.

*Illustration VII. Intelligence and Conscientiousness.* (*Girls.*)

| Conscientiousness. | Intelligence. Slow Intelligent and below | Intelligence. Intelligent and above | Totals |
|---|---|---|---|
| Dull and Moderate | 250 | 67 | 317 |
| Keen ... ... | 76 | 96 | 172 |
| Totals | 326 | 163 | 489 |

Here $\frac{1}{2}(1-\alpha_h) = \cdot 333{,}333$, $\frac{1}{2}(1-\alpha_k) = \cdot 351{,}738$,

and $d/N = \cdot 196{,}3190$.

Hence $h = \cdot 43072$, $k = \cdot 38063$.

We have $\theta = \cdot 3072$, $\phi = \cdot 6928$, $\chi = \cdot 8063$, $\psi = \cdot 1937$.

Thus
$$z_{\theta,\chi} = \cdot 1341{,}9536 \begin{Bmatrix} \cdot 206{,}888 \\ \cdot 215{,}474 \end{Bmatrix} + \cdot 0595{,}0464 \begin{Bmatrix} \cdot 190{,}114 \\ \cdot 198{,}360 \end{Bmatrix}$$
$$+ \cdot 5586{,}0464 \begin{Bmatrix} \cdot 191{,}979 \\ \cdot 200{,}401 \end{Bmatrix} + \cdot 2476{,}9536 \begin{Bmatrix} \cdot 176{,}847 \\ \cdot 184{,}994 \end{Bmatrix}.$$

The observed value of $d/N$ occurs in the $h = \cdot 4$ to $\cdot 5$ and $k = \cdot 3$ to $\cdot 4$ in the $r$ tables for $\cdot 45$, $\cdot 50$ and $\cdot 55$; we therefore try $r = \cdot 50$ first, i.e. the upper numbers in the curled brackets, and find $d/N = \cdot 190{,}1206$ too small; we then compute for the higher number $r = \cdot 55$, i.e. the lower figures. These give us $d/N = \cdot 198{,}4860$. Interpolating linearly for $d/N = \cdot 196{,}3190$, we have

$$r = \cdot 50 + \cdot 05 \times \frac{61934}{83654} = \cdot 5370.$$

The value found from the full tetrachoric equation is $r = \cdot 5370$, in exact agreement. Thus the hyperbolic formula alone appears adequate in this case.

*Illustration VIII. Habits of Mother and Cleanliness of Home.* (*Bradford.*)

Habits of Mother.

| Cleanliness of Home. | Good | Indifferent and Bad | Totals |
|---|---|---|---|
| Clean ... ... | 1009 | 164 | 1173 |
| Fair and Dirty | 144 | 379 | 523 |
| Totals | 1153 | 543 | 1696 |

Here $\frac{1}{2}(1-\alpha_h) = \cdot 320{,}165$, $\frac{1}{2}(1-\alpha_k) = \cdot 308{,}373$,

and $d/N = \cdot 223{,}4670$.

Hence $h = \cdot 46724$, $k = \cdot 50047$.

We have $\theta = \cdot 6724$, $\phi = \cdot 3276$, $\chi = \cdot 0047$, $\psi = \cdot 9953$.

The value of $d/N$ occurs only in the $(h, k)$ range for $r = \cdot 80$ and $r = \cdot 85$. It is only necessary therefore to interpolate for these tables. We have

$$z_{\theta,\chi} = \cdot 3260{,}6028 \begin{Bmatrix} \cdot 233{,}252 \\ \cdot 245{,}704 \end{Bmatrix} + \cdot 6692{,}3972 \begin{Bmatrix} \cdot 218{,}560 \\ \cdot 230{,}869 \end{Bmatrix}$$
$$+ \cdot 0015{,}3972 \begin{Bmatrix} \cdot 215{,}207 \\ \cdot 226{,}826 \end{Bmatrix} + \cdot 0031{,}6028 \begin{Bmatrix} \cdot 202{,}791 \\ \cdot 214{,}580 \end{Bmatrix}$$
$$= \begin{Bmatrix} \cdot 223{,}2955 \\ \cdot 235{,}6484 \end{Bmatrix}.$$

Hence, interpolating for the observed value of $d/N$, we find

$$r = \cdot 80 + \cdot 05 \times \frac{1715}{123{,}529} = \cdot 8007.$$

Using 18 tetrachoric functions and adjusting for the remainder we find $r = \cdot 80013$.

The labour of determining an equation of this order is very great, and we cannot even then be absolutely sure to a unit in the last figure of $r$. We accordingly proceeded to a more close interpolation from the tables. We need the following $\delta^2$'s and $\delta'^2$'s:

| | $r = \cdot 80$ | $r = \cdot 85$ |
|---|---|---|
| $\delta^2 z_{0,0}$ | $-\cdot 001{,}305$ | $-\cdot 001{,}695$ |
| $\delta^2 z_{1,0}$ | $-\cdot 001{,}077$ | $-\cdot 001{,}454$ |
| $\delta^2 z_{0,1}$ | $-\cdot 001{,}279$ | $-\cdot 001{,}633$ |
| $\delta^2 z_{1,1}$ | $-\cdot 001{,}116$ | $-\cdot 001{,}486$ |
| $\delta'^2 z_{0,0}$ | $-\cdot 000{,}989$ | $-\cdot 001{,}343$ |
| $\delta'^2 z_{1,0}$ | $-\cdot 001{,}077$ | $-\cdot 001{,}454$ |
| $\delta'^2 z_{0,1}$ | $-\cdot 000{,}648$ | $-\cdot 000{,}930$ |
| $\delta'^2 z_{1,1}$ | $-\cdot 000{,}795$ | $-\cdot 001{,}125$ |

Using formula ($\beta$) we have for the remainder

$$\begin{aligned} R = &-\tfrac{1}{6}(\cdot 6724 \times \cdot 3276) \\ &-\cdot 0367{,}1304 \end{aligned} \Big] \Big[ 1\cdot 3276 \left( \cdot 9953 \begin{Bmatrix} -\cdot 001{,}305 \\ -\cdot 001{,}695 \end{Bmatrix} + \cdot 0047 \begin{Bmatrix} -\cdot 001{,}279 \\ -\cdot 001{,}633 \end{Bmatrix} \right) + 1\cdot 6724 \left( \cdot 9953 \begin{Bmatrix} -\cdot 001{,}077 \\ -\cdot 001{,}454 \end{Bmatrix} + \cdot 0047 \begin{Bmatrix} -\cdot 001{,}116 \\ -\cdot 001{,}486 \end{Bmatrix} \right) \Big]$$

$$\begin{aligned} &-\tfrac{1}{6}(\cdot 0047 \times \cdot 9953) \\ &-\cdot 0007{,}7965 \end{aligned} \Big] \Big[ 1\cdot 9953 \left( \cdot 3276 \begin{Bmatrix} -\cdot 000{,}989 \\ -\cdot 001{,}343 \end{Bmatrix} + \cdot 6724 \begin{Bmatrix} -\cdot 001{,}077 \\ -\cdot 001{,}454 \end{Bmatrix} \right) + 1\cdot 0047 \left( \cdot 3276 \begin{Bmatrix} -\cdot 000{,}648 \\ -\cdot 000{,}930 \end{Bmatrix} + \cdot 6724 \begin{Bmatrix} -\cdot 000{,}795 \\ -\cdot 001{,}125 \end{Bmatrix} \right) \Big],$$

or
$$\begin{aligned} R &= +\cdot 000{,}1320, \text{ for } r = \cdot 80, \\ &= +\cdot 000{,}1749, \text{ for } r = \cdot 85. \end{aligned}$$

Hence
$$\begin{aligned} d/N &= \cdot 223{,}4275, \text{ for } r = \cdot 80, \\ &= \cdot 235{,}9233, \text{ for } r = \cdot 85. \end{aligned}$$

Then interpolating linearly for $d/N = \cdot 223{,}4670$, we have

$$r = \cdot 80 + \cdot 05 \times \frac{395}{124958} = \cdot 80016,$$

which is adequately close to the result ·80013 of the equation method.

The forward difference formula ($\epsilon$) gives a slightly worse result, namely $r = \cdot 80021$.

*Illustration IX. Binocular Vision and Vision of Better Eye.* (*Boys.*)

| Binocular Vision. | Vision of Better Eye. 6/9 and better | Below 6/9 | Totals |
|---|---|---|---|
| 6/6 and better | 296 | 9 | 305 |
| Below 6/6 ... | 35 | 99 | 134 |
| Totals | 331 | 108 | 439 |

When the correlation is over ·90 it is hard to test the accuracy of the result by the complete tetrachoric equation; the terms converge so slowly that it is practically impossible to compute enough of them. On the other hand, if we take material of which the correlation is known, say by the product-moment method, it is again not easy to obtain adequate numbers in the cells off the "correlation diagonal." The above table is one for the correlation of binocular vision and the vision of the better eye. The correlation coefficient worked by the product-moment process on the full table was ·9485. The above was the best fourfold table we could make, i.e. the others gave two or three units only in one or other cell.

We have $\frac{1}{2}(1-\alpha_h) = \cdot 246{,}0136$, $\frac{1}{2}(1-\alpha_k) = \cdot 305{,}2391$,

and $d/N = \cdot 225{,}5125$.

Hence we have $h = \cdot 68709$, $k = \cdot 50939$.

Further $\theta = \cdot 8709$, $\phi = \cdot 1291$, $\chi = \cdot 0939$, $\psi = \cdot 9061$.

The value required of $d/N$ occurs within the given $(h, k)$ range only for $r = \cdot 95$ and $r = \cdot 90$. Accordingly we have

$$z_{\theta,\chi} = \cdot 1169{,}7751 \begin{Bmatrix} \cdot 245{,}878 \\ \cdot 228{,}353 \end{Bmatrix} + \cdot 7891{,}2249 \begin{Bmatrix} \cdot 225{,}035 \\ \cdot 209{,}730 \end{Bmatrix}$$
$$+ \cdot 0121{,}2249 \begin{Bmatrix} \cdot 232{,}102 \\ \cdot 214{,}481 \end{Bmatrix} + \cdot 0817{,}7751 \begin{Bmatrix} \cdot 215{,}259 \\ \cdot 198{,}793 \end{Bmatrix},$$

or
$$d/N = \cdot 226{,}759, \quad \text{for } r = \cdot 95,$$
$$= \cdot 211{,}072, \quad \text{for } r = \cdot 90.$$

Interpolating linearly for $d/N = \cdot 225{,}513$, we have

$$r = \cdot 90 + \cdot 05 \times \frac{14446}{15687} = \cdot 9460.$$

This is as good a result as could possibly be anticipated for the agreement of a product-moment and a tetrachoric coefficient, when we bear in mind that the accordance depends on the existence of nine individuals only in the right-hand upper cell.

To sum up we see that the new table throughout the range of $r$ gives very satisfactory results with far less labour than the equation method, if we use second differences for $h$ and $k$ and linear interpolation for $r$. There is not much difference in accuracy between applying a central difference and a forward difference second order interpolation. For many purposes where two to three decimals are adequate in the value of $r$ the simple hyperbolic formula ($\alpha$) is sufficient.

We will now proceed to illustrate the application of the present table to determine the cell contents of a contingency table of which the frequency distribution is supposed to be normal, and it is desired to test the accuracy of the hypothesis.

Unfortunately until the second series of tables is completed, namely those for $r$ negative and $h$ and $k$ of the same sign, it is not possible to work out all the cell contents of a contingency table by aid of the present short method. It is necessary to work out a certain number of them still by aid of the tetrachoric expansion. The reader who has this to do will rapidly appreciate how much the present table alone saves him.

*Illustration X.* To test whether the following contingency table for Stature in Father and Son may reasonably be considered normal.

Stature of Father.

| Stature of Son. | Short | Shortish | Medium | Tallish | Tall | Totals |
|---|---|---|---|---|---|---|
| Short | 203* | 91 | 26† | 9† | 6† | 335 |
| Shortish | 95 | 75 | 66 | 22 | 26 | 284 |
| Medium | 30† | 36 | 37* | 14* | 20* | 137 |
| Tallish | 18† | 27 | 26* | 11* | 23* | 105 |
| Tall | 12† | 35 | 25* | 13* | 54* | 139 |
| Totals | 358 | 264 | 180 | 69 | 129 | 1000 |

This table has been reduced to broad categories from a much larger table where the categories were all quantitatively given. The ten cell contents in which an asterisk is placed were found with great ease by means of formula ($\alpha$). The contents of six further cells—those in which a dagger is placed—were found far less easily by aid of tetrachoric expansions. The contents of the remaining nine cells could then be found from the marginal totals and a knowledge of the sixteen cell contents already found.

The actual correlation coefficient of this table as found by the product-moment method on the original data was $r = \cdot5189$. The result would have been practically the same had the value obtained by mean square contingency corrected for class indices, namely $r = \cdot5179$, been used. The following table gives the computed frequencies:

Stature of Father.

| Stature of Son. | Short | Shortish | Medium | Tallish | Tall | Totals |
|---|---|---|---|---|---|---|
| Short | 196·4 | 83·4 | 36·2 | 9·4 | 9·6 | 335 |
| Shortish | 99·3 | 86·5 | 54·6 | 18·5 | 25·1 | 284 |
| Medium | 32·2 | 39·9 | 31·4 | 12·5 | 21·0 | 137 |
| Tallish | 17·8 | 22·4 | 27·6 | 12·3 | 24·9 | 105 |
| Tall | 12·3 | 31·8 | 30·2 | 16·3 | 48·4 | 139 |
| Totals | 358 | 264 | 180 | 69 | 129 | 1000 |

The method of determining the cell contents from the table is easily indicated. Take for example the cell for "Tall" Sons and "Tall" Fathers. We have

$$\tfrac{1}{2}(1-\alpha_h) = \cdot129, \quad \tfrac{1}{2}(1-\alpha_k) = \cdot139,$$

whence we have $h = 1\cdot13113$ and $k = 1\cdot12639$. Thus

$$\theta = \cdot3113, \quad \phi = \cdot6887, \quad \chi = \cdot2639, \quad \text{and} \quad \psi = \cdot7361.$$

Here we find for

$$r = \cdot 50, \quad d/N = \cdot 046{,}831\,; \quad r = \cdot 55, \quad d/N = \cdot 050880,$$

or by interpolation for $r = \cdot 5189$ we have

$$d/N = \cdot 04836, \quad \text{and} \quad d = 48 \cdot 4.$$

Next we take "Tall" Fathers with "Tall" and "Tallish" Sons and find in the same way $d = 73 \cdot 3$; the difference of this and the former $d$ gives $24 \cdot 9$ as the cell frequency of "Tall" Fathers and "Tallish" Sons. Thus gradually the individual cell frequency is built up from combinations of cell frequencies.

In the cases where we still had to determine cell frequencies from tetrachoric series, two sets of ten tetrachoric functions had to be determined from the tables of these functions; then the product of these functions, pair and pair was taken, and these products again had to be multiplied by the respective powers of $r$ from 1 to 10. The series thus formed, i.e.

$$\frac{d}{N} = \tau_0\tau_0' + r\tau_1\tau_1' + r^2\tau_2\tau_2' + \ldots + r^{10}\tau_{10}\tau_{10}',$$

gives the required cell content. The process is a very laborious one, and we can only hope to avoid it completely when the second series of these tables is computed; it will then be exceedingly easy to determine the cell contents of any contingency table on the hypothesis that it is a normal distribution.

It seemed worth while applying the goodness of fit test. Here, as we have applied the normal surface *assuming the marginal frequencies to be the same for the surface and the sample*, we have introduced ten restrictions instead of the single usual one, accordingly we must look up $P$ for $n = 25 - 9 = 16$. The $\chi^2 = 15 \cdot 566$ and we have $P = \cdot 412$, a very reasonable fit. Thus the distribution of Stature in Fathers and Sons may be described legitimately by a normal surface.

| $k$ | $r=\cdot00$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·250000 | ·230086 | ·210370 | ·191044 | ·172289 | ·154269 | ·137127 | ·120982 | ·105928 | $0\cdot0$ |
| $0\cdot1$ | ·230086 | ·211758 | ·193613 | ·175827 | ·158565 | ·141980 | ·126204 | ·111345 | ·097490 | $0\cdot1$ |
| $0\cdot2$ | ·210370 | ·193613 | ·177022 | ·160760 | ·144978 | ·129814 | ·115389 | ·101804 | ·089136 | $0\cdot2$ |
| $0\cdot3$ | ·191044 | ·175827 | ·160760 | ·145992 | ·131659 | ·117889 | ·104789 | ·092452 | ·080948 | $0\cdot3$ |
| $0\cdot4$ | ·172289 | ·158565 | ·144978 | ·131659 | ·118734 | ·106315 | ·094502 | ·083375 | ·073001 | $0\cdot4$ |
| $0\cdot5$ | ·154269 | ·141980 | ·129814 | ·117889 | ·106315 | ·095195 | ·084617 | ·074655 | ·065365 | $0\cdot5$ |
| $0\cdot6$ | ·137127 | ·126204 | ·115389 | ·104789 | ·094502 | ·084617 | ·075215 | ·066359 | ·058102 | $0\cdot6$ |
| $0\cdot7$ | ·120982 | ·111345 | ·101804 | ·092452 | ·083375 | ·074655 | ·066359 | ·058546 | ·051261 | $0\cdot7$ |
| $0\cdot8$ | ·105928 | ·097490 | ·089136 | ·080948 | ·073001 | ·065365 | ·058102 | ·051261 | ·044883 | $0\cdot8$ |
| $0\cdot9$ | ·092030 | ·084699 | ·077442 | ·070327 | ·063423 | ·056789 | ·050479 | ·044536 | ·038994 | $0\cdot9$ |
| $1\cdot0$ | ·079328 | ·073009 | ·066753 | ·060620 | ·054669 | ·048951 | ·043512 | ·038389 | ·033612 | $1\cdot0$ |
| $1\cdot1$ | ·067833 | ·062430 | ·057080 | ·051836 | ·046748 | ·041858 | ·037207 | ·032826 | ·028742 | $1\cdot1$ |
| $1\cdot2$ | ·057535 | ·052952 | ·048414 | ·043967 | ·039650 | ·035503 | ·031558 | ·027843 | ·024378 | $1\cdot2$ |
| $1\cdot3$ | ·048400 | ·044545 | ·040728 | ·036986 | ·033355 | ·029867 | ·026548 | ·023422 | ·020508 | $1\cdot3$ |
| $1\cdot4$ | ·040378 | ·037162 | ·033978 | ·030856 | ·027827 | ·024916 | ·022148 | ·019540 | ·017109 | $1\cdot4$ |
| $1\cdot5$ | ·033404 | ·030743 | ·028108 | ·025526 | ·023020 | ·020613 | ·018322 | ·016165 | ·014153 | $1\cdot5$ |
| $1\cdot6$ | ·027400 | ·025217 | ·023056 | ·020938 | ·018883 | ·016908 | ·015029 | ·013259 | ·011610 | $1\cdot6$ |
| $1\cdot7$ | ·022283 | ·020508 | ·018750 | ·017028 | ·015356 | ·013750 | ·012222 | ·010783 | ·009441 | $1\cdot7$ |
| $1\cdot8$ | ·017965 | ·016534 | ·015117 | ·013729 | ·012381 | ·011086 | ·009854 | ·008694 | ·007612 | $1\cdot8$ |
| $1\cdot9$ | ·014358 | ·013215 | ·012082 | ·010972 | ·009895 | ·008860 | ·007876 | ·006948 | ·006084 | $1\cdot9$ |
| $2\cdot0$ | ·011375 | ·010469 | ·009572 | ·008693 | ·007839 | ·007019 | ·006239 | ·005505 | ·004820 | $2\cdot0$ |
| $2\cdot1$ | ·008932 | ·008221 | ·007516 | ·006826 | ·006156 | ·005512 | ·004899 | ·004323 | ·003785 | $2\cdot1$ |
| $2\cdot2$ | ·006952 | ·006398 | ·005850 | ·005312 | ·004791 | ·004290 | ·003813 | ·003364 | ·002946 | $2\cdot2$ |
| $2\cdot3$ | ·005362 | ·004935 | ·004512 | ·004098 | ·003695 | ·003309 | ·002941 | ·002595 | ·002272 | $2\cdot3$ |
| $2\cdot4$ | ·004099 | ·003772 | ·003449 | ·003132 | ·002825 | ·002529 | ·002248 | ·001984 | ·001737 | $2\cdot4$ |
| $2\cdot5$ | ·003105 | ·002858 | ·002613 | ·002373 | ·002140 | ·001916 | ·001703 | ·001503 | ·001316 | $2\cdot5$ |
| $2\cdot6$ | ·002331 | ·002145 | ·001961 | ·001781 | ·001606 | ·001438 | ·001278 | ·001128 | ·000987 | $2\cdot6$ |

| $k$ | $r=\cdot00$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·092030 | ·079328 | ·067833 | ·057535 | ·048400 | ·040378 | ·033404 | ·027400 | ·022283 | $0\cdot0$ |
| $0\cdot1$ | ·084699 | ·073009 | ·062430 | ·052952 | ·044545 | ·037162 | ·030743 | ·025217 | ·020508 | $0\cdot1$ |
| $0\cdot2$ | ·077442 | ·066753 | ·057080 | ·048414 | ·040728 | ·033978 | ·028108 | ·023056 | ·018750 | $0\cdot2$ |
| $0\cdot3$ | ·070327 | ·060620 | ·051836 | ·043967 | ·036986 | ·030856 | ·025526 | ·020938 | ·017028 | $0\cdot3$ |
| $0\cdot4$ | ·063423 | ·054669 | ·046748 | ·039650 | ·033355 | ·027827 | ·023020 | ·018883 | ·015356 | $0\cdot4$ |
| $0\cdot5$ | ·056789 | ·048951 | ·041858 | ·035503 | ·029867 | ·024916 | ·020613 | ·016908 | ·013750 | $0\cdot5$ |
| $0\cdot6$ | ·050479 | ·043512 | ·037207 | ·031558 | ·026548 | ·022148 | ·018322 | ·015029 | ·012222 | $0\cdot6$ |
| $0\cdot7$ | ·044536 | ·038389 | ·032826 | ·027843 | ·023422 | ·019540 | ·016165 | ·013259 | ·010783 | $0\cdot7$ |
| $0\cdot8$ | ·038994 | ·033612 | ·028742 | ·024378 | ·020508 | ·017109 | ·014153 | ·011610 | ·009441 | $0\cdot8$ |
| $0\cdot9$ | ·033878 | ·029202 | ·024971 | ·021180 | ·017817 | ·014864 | ·012297 | ·010086 | ·008203 | $0\cdot9$ |
| $1\cdot0$ | ·029202 | ·025171 | ·021524 | ·018256 | ·015358 | ·012812 | ·010599 | ·008694 | ·007071 | $1\cdot0$ |
| $1\cdot1$ | ·024971 | ·021524 | ·018405 | ·015611 | ·013133 | ·010956 | ·009063 | ·007434 | ·006046 | $1\cdot1$ |
| $1\cdot2$ | ·021180 | ·018256 | ·015611 | ·013241 | ·011139 | ·009293 | ·007688 | ·006306 | ·005128 | $1\cdot2$ |
| $1\cdot3$ | ·017817 | ·015358 | ·013133 | ·011129 | ·009370 | ·007817 | ·006467 | ·005305 | ·004314 | $1\cdot3$ |
| $1\cdot4$ | ·014864 | ·012812 | ·010956 | ·009293 | ·007817 | ·006522 | ·005395 | ·004425 | ·003599 | $1\cdot4$ |
| $1\cdot5$ | ·012297 | ·010599 | ·009063 | ·007688 | ·006467 | ·005395 | ·004463 | ·003661 | ·002977 | $1\cdot5$ |
| $1\cdot6$ | ·010086 | ·008694 | ·007434 | ·006306 | ·005305 | ·004425 | ·003661 | ·003003 | ·002442 | $1\cdot6$ |
| $1\cdot7$ | ·008203 | ·007071 | ·006046 | ·005128 | ·004314 | ·003599 | ·002977 | ·002442 | ·001986 | $1\cdot7$ |
| $1\cdot8$ | ·006613 | ·005700 | ·004875 | ·004134 | ·003478 | ·002902 | ·002400 | ·001969 | ·001601 | $1\cdot8$ |
| $1\cdot9$ | ·005286 | ·004556 | ·003896 | ·003304 | ·002780 | ·002319 | ·001918 | ·001574 | ·001280 | $1\cdot9$ |
| $2\cdot0$ | ·004187 | ·003609 | ·003086 | ·002618 | ·002202 | ·001837 | ·001520 | ·001247 | ·001014 | $2\cdot0$ |
| $2\cdot1$ | ·003288 | ·002834 | ·002424 | ·002056 | ·001729 | ·001443 | ·001194 | ·000979 | ·000796 | $2\cdot1$ |
| $2\cdot2$ | ·002559 | ·002206 | ·001886 | ·001600 | ·001346 | ·001123 | ·000929 | ·000762 | ·000620 | $2\cdot2$ |
| $2\cdot3$ | ·001974 | ·001701 | ·001455 | ·001234 | ·001038 | ·000866 | ·000716 | ·000588 | ·000478 | $2\cdot3$ |
| $2\cdot4$ | ·001509 | ·001301 | ·001112 | ·000943 | ·000794 | ·000662 | ·000548 | ·000449 | ·000365 | $2\cdot4$ |
| $2\cdot5$ | ·001143 | ·000985 | ·000842 | ·000715 | ·000601 | ·000502 | ·000415 | ·000340 | ·000277 | $2\cdot5$ |
| $2\cdot6$ | ·000858 | ·000740 | ·000632 | ·000536 | ·000451 | ·000376 | ·000311 | ·000255 | ·000208 | $2\cdot6$ |

| $k$ | $r=\cdot00$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·017965 | ·014358 | ·011375 | ·008932 | ·006952 | ·005362 | ·004099 | ·003105 | ·002331 | 0·0 |
| 0·1 | ·016534 | ·013215 | ·010469 | ·008221 | ·006398 | ·004935 | ·003772 | ·002858 | ·002145 | 0·1 |
| 0·2 | ·015117 | ·012082 | ·009572 | ·007516 | ·005850 | ·004512 | ·003449 | ·002613 | ·001961 | 0·2 |
| 0·3 | ·013729 | ·010972 | ·008693 | 006826 | ·005312 | ·004098 | ·003132 | ·002373 | ·001781 | 0·3 |
| 0·4 | ·012381 | ·009895 | ·007839 | ·006156 | ·004791 | ·003695 | ·002825 | ·002140 | ·001606 | 0·4 |
| 0·5 | ·011086 | ·008860 | ·007019 | ·005512 | ·004290 | ·003309 | ·002529 | ·001916 | ·001438 | 0·5 |
| 0·6 | ·009854 | ·007876 | ·006239 | ·004899 | ·003813 | ·002941 | ·002248 | ·001703 | ·001278 | 0·6 |
| 0·7 | ·008694 | ·006948 | ·005505 | ·004323 | ·003364 | ·002595 | ·001984 | ·001503 | ·001128 | 0·7 |
| 0·8 | ·007612 | ·006084 | ·004820 | ·003785 | ·002946 | ·002272 | ·001737 | ·001316 | ·000987 | 0·8 |
| 0·9 | ·006613 | ·005286 | ·004187 | ·003288 | ·002559 | ·001974 | ·001509 | ·001143 | ·000858 | 0·9 |
| 1·0 | ·005709 | ·004556 | ·003609 | ·002834 | ·002206 | ·001701 | ·001301 | ·000985 | ·000740 | 1·0 |
| 1·1 | ·004875 | ·003896 | ·003086 | ·002424 | ·001886 | ·001455 | ·001112 | ·000842 | ·000632 | 1·1 |
| 1·2 | ·004134 | ·003304 | ·002618 | ·002056 | ·001600 | ·001234 | ·000943 | ·000715 | ·000536 | 1·2 |
| 1·3 | ·003478 | ·002780 | ·002202 | ·001729 | ·001346 | ·001038 | ·000794 | ·000601 | ·000451 | 1·3 |
| 1·4 | ·002902 | ·002319 | ·001837 | ·001443 | ·001123 | ·000866 | ·000662 | ·000502 | ·000376 | 1·4 |
| 1·5 | ·002400 | ·001918 | ·001520 | ·001194 | ·000929 | ·000716 | ·000548 | ·000415 | ·000311 | 1·5 |
| 1·6 | ·001969 | ·001574 | ·001247 | ·000979 | ·000762 | ·000588 | ·000449 | ·000340 | ·000255 | 1·6 |
| 1·7 | ·001601 | ·001280 | ·001014 | ·000796 | ·000620 | ·000478 | ·000365 | ·000277 | ·000208 | 1·7 |
| 1·8 | ·001291 | ·001032 | ·000817 | ·000642 | ·000500 | ·000385 | ·000295 | ·000223 | ·000168 | 1·8 |
| 1·9 | ·001032 | ·000825 | ·000653 | ·000513 | ·000399 | ·000308 | ·000235 | ·000178 | ·000134 | 1·9 |
| 2·0 | ·000817 | ·000653 | ·000518 | ·000406 | ·000316 | ·000244 | ·000186 | ·000141 | ·000106 | 2·0 |
| 2·1 | ·000642 | ·000513 | ·000406 | ·000319 | ·000248 | ·000192 | ·000146 | ·000111 | ·000083 | 2·1 |
| 2·2 | ·000500 | ·000399 | ·000316 | ·000248 | ·000193 | ·000149 | ·000114 | ·000086 | ·000065 | 2·2 |
| 2·3 | ·000385 | ·000308 | ·000244 | ·000192 | ·000149 | ·000115 | ·000088 | ·000067 | ·000050 | 2·3 |
| 2·4 | ·000295 | ·000235 | ·000186 | ·000146 | ·000114 | ·000088 | ·000067 | ·000051 | ·000038 | 2·4 |
| 2·5 | ·000223 | ·000178 | ·000141 | ·000111 | ·000086 | ·000067 | ·000051 | ·000039 | ·000029 | 2·5 |
| 2·6 | ·000168 | ·000134 | ·000106 | ·000083 | ·000065 | ·000050 | ·000038 | ·000029 | ·000022 | 2·6 |

| $k$ | $r=\cdot05$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·257961 | ·238007 | ·218173 | ·198655 | ·179638 | ·161294 | ·143775 | ·127212 | ·111707 | 0·0 |
| 0·1 | ·238007 | ·219642 | ·201381 | ·183405 | ·165885 | ·148979 | ·132829 | ·117555 | ·103252 | 0·1 |
| 0·2 | ·218173 | ·201381 | ·184679 | ·168231 | ·152195 | ·136717 | ·121926 | ·107932 | ·094824 | 0·2 |
| 0·3 | ·198655 | ·183405 | ·168231 | ·153283 | ·138706 | ·124630 | ·111174 | ·098438 | ·086506 | 0·3 |
| 0·4 | ·179638 | ·165885 | ·152195 | ·138706 | ·125545 | ·112832 | ·100676 | ·089166 | ·078378 | 0·4 |
| 0·5 | ·161294 | ·148979 | ·136717 | ·124630 | ·112832 | ·101433 | ·090528 | ·080201 | ·070517 | 0·5 |
| 0·6 | ·143775 | ·132829 | ·121926 | ·111174 | ·100676 | ·090528 | ·080818 | ·071617 | ·062987 | 0·6 |
| 0·7 | ·127212 | ·117555 | ·107932 | ·098438 | ·089166 | ·080201 | ·071617 | ·063482 | ·055848 | 0·7 |
| 0·8 | ·111707 | ·103252 | ·094824 | ·086506 | ·078378 | ·070517 | ·062987 | ·055848 | ·049146 | 0·8 |
| 0·9 | ·097338 | ·089993 | ·082668 | ·075436 | ·068367 | ·061527 | ·054973 | ·048756 | ·042918 | 0·9 |
| 1·0 | ·084154 | ·077823 | ·071507 | ·065269 | ·059169 | ·053264 | ·047604 | ·042233 | ·037187 | 1·0 |
| 1·1 | ·072178 | ·066765 | ·061363 | ·056025 | ·050803 | ·045746 | ·040896 | ·036293 | ·031967 | 1·1 |
| 1·2 | ·061408 | ·056817 | ·052233 | ·047703 | ·043269 | ·038972 | ·034851 | ·030938 | ·027258 | 1·2 |
| 1·3 | ·051818 | ·047956 | ·044099 | ·040285 | ·036551 | ·032932 | ·029458 | ·026158 | ·023054 | 1·3 |
| 1·4 | ·043364 | ·040143 | ·036925 | ·033740 | ·030622 | ·027597 | ·024694 | ·021935 | ·019338 | 1·4 |
| 1·5 | ·035986 | ·033322 | ·030659 | ·028023 | ·025440 | ·022934 | ·020528 | ·018240 | ·016085 | 1·5 |
| 1·6 | ·029611 | ·027426 | ·025241 | ·023078 | ·020957 | ·018898 | ·016921 | ·015039 | ·013267 | 1·6 |
| 1·7 | ·024157 | ·022381 | ·020604 | ·018843 | ·017116 | ·015440 | ·013828 | ·012295 | ·010850 | 1·7 |
| 1·8 | ·019539 | ·018107 | ·016673 | ·015253 | ·013860 | ·012506 | ·011204 | ·009965 | ·008796 | 1·8 |
| 1·9 | ·015666 | ·014522 | ·013376 | ·012240 | ·011125 | ·010042 | ·008999 | ·008006 | ·007070 | 1·9 |
| 2·0 | ·012451 | ·011545 | ·010637 | ·009736 | ·008852 | ·007993 | ·007165 | ·006377 | ·005633 | 2·0 |
| 2·1 | ·009808 | ·009097 | ·008384 | ·007677 | ·006982 | ·006306 | ·005655 | ·005034 | ·004448 | 2·1 |
| 2·2 | ·007658 | ·007105 | ·006550 | ·005999 | ·005458 | ·004931 | ·004423 | ·003939 | ·003482 | 2·2 |
| 2·3 | ·005926 | ·005499 | ·005071 | ·004646 | ·004228 | ·003821 | ·003429 | ·003054 | ·002701 | 2·3 |
| 2·4 | ·004545 | ·004219 | ·003891 | ·003566 | ·003246 | ·002935 | ·002634 | ·002348 | ·002076 | 2·4 |
| 2·5 | ·003454 | ·003207 | ·002959 | ·002713 | ·002470 | ·002234 | ·002006 | ·001788 | ·001582 | 2·5 |
| 2·6 | ·002601 | ·002416 | ·002230 | ·002044 | ·001862 | ·001685 | ·001513 | ·001349 | ·001194 | 2·6 |

| $k$ | $r=\cdot05$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·097338 | ·084154 | ·072178 | ·061408 | ·051818 | ·043364 | ·035986 | ·029611 | ·024157 | 0·0 |
| 0·1 | ·089993 | ·077823 | ·066765 | ·056817 | ·047956 | ·040143 | ·033322 | ·027426 | ·022381 | 0·1 |
| 0·2 | ·082668 | ·071507 | ·061363 | ·052233 | ·044099 | ·036925 | ·030659 | ·025241 | ·020604 | 0·2 |
| 0·3 | ·075436 | ·065269 | ·056025 | ·047703 | ·040285 | ·033740 | ·028023 | ·023078 | ·018843 | 0·3 |
| 0·4 | ·068367 | ·059169 | ·050803 | ·043269 | ·036551 | ·030622 | ·025440 | ·020957 | ·017116 | 0·4 |
| 0·5 | ·061527 | ·053264 | ·045746 | ·038972 | ·032932 | ·027597 | ·022934 | ·018898 | ·015440 | 0·5 |
| 0·6 | ·054973 | ·047604 | ·040896 | ·034851 | ·029458 | ·024694 | ·020528 | ·016921 | ·013828 | 0·6 |
| 0·7 | ·048756 | ·042233 | ·036293 | ·030938 | ·026158 | ·021935 | ·018240 | ·015039 | ·012295 | 0·7 |
| 0·8 | ·042918 | ·037187 | ·031967 | ·027258 | ·023054 | ·019338 | ·016085 | ·013267 | ·010850 | 0·8 |
| 0·9 | ·037490 | ·032494 | ·027941 | ·023833 | ·020164 | ·016919 | ·014078 | ·011615 | ·009502 | 0·9 |
| 1·0 | ·032494 | ·028172 | ·024232 | ·020676 | ·017499 | ·014687 | ·012225 | ·010090 | ·008257 | 1·0 |
| 1·1 | ·027941 | ·024232 | ·020850 | ·017796 | ·015066 | ·012650 | ·010533 | ·008696 | ·007119 | 1·1 |
| 1·2 | ·023833 | ·020676 | ·017796 | ·015195 | ·012868 | ·010808 | ·009002 | ·007435 | ·006088 | 1·2 |
| 1·3 | ·020164 | ·017499 | ·015066 | ·012868 | ·010901 | ·009159 | ·007631 | ·006305 | ·005165 | 1·3 |
| 1·4 | ·016919 | ·014687 | ·012650 | ·010808 | ·009159 | ·007698 | ·006416 | ·005303 | ·004345 | 1·4 |
| 1·5 | ·014078 | ·012225 | ·010533 | ·009002 | ·007631 | ·006416 | ·005350 | ·004423 | ·003626 | 1·5 |
| 1·6 | ·011615 | ·010090 | ·008696 | ·007435 | ·006305 | ·005303 | ·004423 | ·003658 | ·003000 | 1·6 |
| 1·7 | ·009502 | ·008257 | ·007119 | ·006088 | ·005165 | ·004345 | ·003626 | ·003000 | ·002461 | 1·7 |
| 1·8 | ·007706 | ·006699 | ·005777 | ·004943 | ·004195 | ·003530 | ·002947 | ·002439 | ·002002 | 1·8 |
| 1·9 | ·006196 | ·005388 | ·004648 | ·003978 | ·003377 | ·002843 | ·002374 | ·001966 | ·001614 | 1·9 |
| 2·0 | ·004938 | ·004295 | ·003707 | ·003174 | ·002695 | ·002270 | ·001896 | ·001571 | ·001290 | 2·0 |
| 2·1 | ·003901 | ·003394 | ·002930 | ·002510 | ·002132 | ·001797 | ·001501 | ·001244 | ·001022 | 2·1 |
| 2·2 | ·003054 | ·002659 | ·002296 | ·001967 | ·001672 | ·001409 | ·001178 | ·000976 | ·000803 | 2·2 |
| 2·3 | ·002370 | ·002064 | ·001783 | ·001528 | ·001299 | ·001096 | ·000916 | ·000760 | ·000625 | 2·3 |
| 2·4 | ·001823 | ·001588 | ·001372 | ·001177 | ·001001 | ·000844 | ·000706 | ·000586 | ·000482 | 2·4 |
| 2·5 | ·001389 | ·001211 | ·001047 | ·000898 | ·000764 | ·000644 | ·000539 | ·000448 | ·000368 | 2·5 |
| 2·6 | ·001049 | ·000915 | ·000791 | ·000679 | ·000578 | ·000488 | ·000408 | ·000339 | ·000279 | 2·6 |

| $k$ | $r=\cdot05$ | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·019539 | ·015666 | ·012451 | ·009808 | ·007658 | ·005926 | ·004545 | ·003454 | ·002601 | 0·0 |
| 0·1 | ·018107 | ·014522 | ·011545 | ·009097 | ·007105 | ·005499 | ·004219 | ·003207 | ·002416 | 0·1 |
| 0·2 | ·016673 | ·013376 | ·010637 | ·008384 | ·006550 | ·005071 | ·003891 | ·002959 | ·002230 | 0·2 |
| 0·3 | ·015253 | ·012240 | ·009736 | ·007677 | ·005999 | ·004646 | ·003566 | ·002713 | ·002044 | 0·3 |
| 0·4 | ·013860 | ·011125 | ·008852 | ·006982 | ·005458 | ·004228 | ·003246 | ·002470 | ·001862 | 0·4 |
| 0·5 | ·012506 | ·010042 | ·007993 | ·006306 | ·004931 | ·003821 | ·002935 | ·002234 | ·001685 | 0·5 |
| 0·6 | ·011204 | ·008999 | ·007165 | ·005655 | ·004423 | ·003429 | ·002634 | ·002006 | ·001513 | 0·6 |
| 0·7 | ·009965 | ·008006 | ·006377 | ·005034 | ·003939 | ·003054 | ·002348 | ·001788 | ·001349 | 0·7 |
| 0·8 | ·008796 | ·007070 | ·005633 | ·004448 | ·003482 | ·002701 | ·002076 | ·001582 | ·001194 | 0·8 |
| 0·9 | ·007706 | ·006196 | ·004938 | ·003901 | ·003054 | ·002370 | ·001823 | ·001389 | ·001049 | 0·9 |
| 1·0 | ·006699 | ·005388 | ·004295 | ·003394 | ·002659 | ·002064 | ·001588 | ·001211 | ·000915 | 1·0 |
| 1·1 | ·005777 | ·004648 | ·003707 | ·002930 | ·002296 | ·001783 | ·001372 | ·001047 | ·000791 | 1·1 |
| 1·2 | ·004943 | ·003978 | ·003174 | ·002510 | ·001967 | ·001528 | ·001177 | ·000898 | ·000679 | 1·2 |
| 1·3 | ·004195 | ·003377 | ·002695 | ·002132 | ·001672 | ·001299 | ·001001 | ·000764 | ·000578 | 1·3 |
| 1·4 | ·003530 | ·002843 | ·002270 | ·001797 | ·001409 | ·001096 | ·000844 | ·000644 | ·000488 | 1·4 |
| 1·5 | ·002947 | ·002374 | ·001896 | ·001501 | ·001178 | ·000916 | ·000706 | ·000539 | ·000408 | 1·5 |
| 1·6 | ·002439 | ·001966 | ·001571 | ·001244 | ·000976 | ·000760 | ·000586 | ·000448 | ·000339 | 1·6 |
| 1·7 | ·002002 | ·001614 | ·001290 | ·001022 | ·000803 | ·000625 | ·000482 | ·000368 | ·000279 | 1·7 |
| 1·8 | ·001629 | ·001314 | ·001050 | ·000832 | ·000654 | ·000509 | ·000393 | ·000300 | ·000228 | 1·8 |
| 1·9 | ·001314 | ·001060 | ·000848 | ·000672 | ·000528 | ·000411 | ·000318 | ·000243 | ·000184 | 1·9 |
| 2·0 | ·001050 | ·000848 | ·000678 | ·000538 | ·000423 | ·000330 | ·000255 | ·000195 | ·000148 | 2·0 |
| 2·1 | ·000832 | ·000672 | ·000538 | ·000427 | ·000336 | ·000262 | ·000202 | ·000155 | ·000117 | 2·1 |
| 2·2 | ·000654 | ·000528 | ·000423 | ·000336 | ·000264 | ·000206 | ·000159 | ·000122 | ·000093 | 2·2 |
| 2·3 | ·000509 | ·000411 | ·000330 | ·000262 | ·000206 | ·000161 | ·000124 | ·000095 | ·000072 | 2·3 |
| 2·4 | ·000393 | ·000318 | ·000255 | ·000202 | ·000159 | ·000124 | ·000096 | ·000074 | ·000056 | 2·4 |
| 2·5 | ·000300 | ·000243 | ·000195 | ·000155 | ·000122 | ·000095 | ·000074 | ·000057 | ·000043 | 2·5 |
| 2·6 | ·000228 | ·000184 | ·000148 | ·000117 | ·000093 | ·000072 | ·000056 | ·000043 | ·000033 | 2·6 |

| $k$ | $r=\cdot10$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·265942 | ·245948 | ·225996 | ·206283 | ·187002 | ·168332 | ·150435 | ·133450 | ·117492 | 0·0 |
| 0·1 | ·245948 | ·227549 | ·209176 | ·191012 | ·173234 | ·156008 | ·139485 | ·123794 | ·109042 | 0·1 |
| 0·2 | ·225996 | ·209176 | ·192368 | ·175741 | ·159456 | ·143667 | ·128512 | ·114110 | ·100562 | 0·2 |
| 0·3 | ·206283 | ·191012 | ·175741 | ·160623 | ·145808 | ·131433 | ·117625 | ·104496 | ·092135 | 0·3 |
| 0·4 | ·187002 | ·173234 | ·159456 | ·145808 | ·132422 | ·119425 | ·106933 | ·095045 | ·083846 | 0·4 |
| 0·5 | ·168332 | ·156008 | ·143667 | ·131433 | ·119425 | ·107758 | ·096536 | ·085849 | ·075774 | 0·5 |
| 0·6 | ·150435 | ·139485 | ·128512 | ·117625 | ·106933 | ·096536 | ·086527 | ·076990 | ·067992 | 0·6 |
| 0·7 | ·133450 | ·123794 | ·114110 | ·104496 | ·095045 | ·085849 | ·076990 | ·068541 | ·060564 | 0·7 |
| 0·8 | ·117492 | ·109042 | ·100562 | ·092135 | ·083846 | ·075774 | ·067992 | ·060564 | ·053546 | 0·8 |
| 0·9 | ·102649 | ·095313 | ·087944 | ·080616 | ·073402 | ·066371 | ·059588 | ·053109 | ·046982 | 0·9 |
| 1·0 | ·088981 | ·082662 | ·076310 | ·069988 | ·063759 | ·057684 | ·051818 | ·046210 | ·040903 | 1·0 |
| 1·1 | ·076521 | ·071122 | ·065690 | ·060280 | ·054945 | ·049738 | ·044705 | ·039890 | ·035330 | 1·1 |
| 1·2 | ·065276 | ·060701 | ·056094 | ·051502 | ·046970 | ·042542 | ·038260 | ·034159 | ·030273 | 1·2 |
| 1·3 | ·055229 | ·051384 | ·047509 | ·043643 | ·039825 | ·036091 | ·032477 | ·029014 | ·025729 | 1·3 |
| 1·4 | ·046342 | ·043138 | ·039906 | ·036678 | ·033488 | ·030366 | ·027342 | ·024441 | ·021687 | 1·4 |
| 1·5 | ·038560 | ·035912 | ·033239 | ·030568 | ·027925 | ·025337 | ·022827 | ·020417 | ·018128 | 1·5 |
| 1·6 | ·031813 | ·029644 | ·027452 | ·025260 | ·023089 | ·020962 | ·018896 | ·016913 | ·015026 | 1·6 |
| 1·7 | ·026023 | ·024261 | ·022480 | ·020696 | ·018928 | ·017194 | ·015510 | ·013890 | ·012348 | 1·7 |
| 1·8 | ·021103 | ·019685 | ·018249 | ·016811 | ·015384 | ·013983 | ·012621 | ·011310 | ·010061 | 1·8 |
| 1·9 | ·016965 | ·015833 | ·014686 | ·013536 | ·012394 | ·011272 | ·010180 | ·009129 | ·008126 | 1·9 |
| 2·0 | ·013518 | ·012623 | ·011715 | ·010804 | ·009898 | ·009008 | ·008140 | ·007304 | ·006506 | 2·0 |
| 2·1 | ·010677 | ·009975 | ·009263 | ·008547 | ·007836 | ·007135 | ·006452 | ·005793 | ·005163 | 2·1 |
| 2·2 | ·008358 | ·007813 | ·007259 | ·006702 | ·006147 | ·005601 | ·005068 | ·004553 | ·004061 | 2·2 |
| 2·3 | ·006484 | ·006064 | ·005638 | ·005208 | ·004780 | ·004358 | ·003946 | ·003547 | ·003166 | 2·3 |
| 2·4 | ·004985 | ·004665 | ·004339 | ·004011 | ·003683 | ·003360 | ·003044 | ·002738 | ·002446 | 2·4 |
| 2·5 | ·003798 | ·003556 | ·003309 | ·003061 | ·002813 | ·002567 | ·002327 | ·002095 | ·001872 | 2·5 |
| 2·6 | ·002867 | ·002686 | ·002501 | ·002315 | ·002128 | ·001944 | ·001763 | ·001588 | ·001420 | 2·6 |

| $k$ | $r=\cdot10$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·102649 | ·088981 | ·076521 | ·065276 | ·055229 | ·046342 | ·038560 | ·031813 | ·026023 | 0·0 |
| 0·1 | ·095313 | ·082662 | ·071122 | ·060701 | ·051384 | ·043138 | ·035912 | ·029644 | ·024261 | 0·1 |
| 0·2 | ·087944 | ·076310 | ·065690 | ·056094 | ·047509 | ·039906 | ·033239 | ·027452 | ·022480 | 0·2 |
| 0·3 | ·080616 | ·069988 | ·060280 | ·051502 | ·043643 | ·036678 | ·030568 | ·025260 | ·020696 | 0·3 |
| 0·4 | ·073402 | ·063759 | ·054945 | ·046970 | ·039825 | ·033488 | ·027925 | ·023089 | ·018928 | 0·4 |
| 0·5 | ·066371 | ·057684 | ·049738 | ·042542 | ·036091 | ·030366 | ·025337 | ·020962 | ·017194 | 0·5 |
| 0·6 | ·059588 | ·051818 | ·044705 | ·038260 | ·032477 | ·027342 | ·022827 | ·018896 | ·015510 | 0·6 |
| 0·7 | ·053109 | ·046210 | ·039890 | ·034159 | ·029014 | ·024441 | ·020417 | ·016913 | ·013890 | 0·7 |
| 0·8 | ·046982 | ·040903 | ·035330 | ·030273 | ·025729 | ·021687 | ·018128 | ·015026 | ·012348 | 0·8 |
| 0·9 | ·041247 | ·035931 | ·031055 | ·026626 | ·022643 | ·019098 | ·015974 | ·013249 | ·010895 | 0·9 |
| 1·0 | ·035931 | ·031320 | ·027086 | ·023238 | ·019775 | ·016690 | ·013969 | ·011593 | ·009540 | 1·0 |
| 1·1 | ·031055 | ·027086 | ·023440 | ·020122 | ·017134 | ·014470 | ·012119 | ·010065 | ·008288 | 1·1 |
| 1·2 | ·026626 | ·023238 | ·020122 | ·017285 | ·014728 | ·012447 | ·010432 | ·008669 | ·007143 | 1·2 |
| 1·3 | ·022643 | ·019775 | ·017134 | ·014728 | ·012558 | ·010620 | ·008906 | ·007407 | ·006107 | 1·3 |
| 1·4 | ·019098 | ·016690 | ·014470 | ·012447 | ·010620 | ·008987 | ·007542 | ·006276 | ·005179 | 1·4 |
| 1·5 | ·015974 | ·013969 | ·012119 | ·010432 | ·008906 | ·007542 | ·006334 | ·005275 | ·004356 | 1·5 |
| 1·6 | ·013249 | ·011593 | ·010065 | ·008669 | ·007407 | ·006276 | ·005275 | ·004396 | ·003632 | 1·6 |
| 1·7 | ·010895 | ·009540 | ·008288 | ·007143 | ·006107 | ·005179 | ·004356 | ·003632 | ·003004 | 1·7 |
| 1·8 | ·008883 | ·007783 | ·006766 | ·005836 | ·004993 | ·004237 | ·003566 | ·002976 | ·002463 | 1·8 |
| 1·9 | ·007179 | ·006294 | ·005476 | ·004726 | ·004046 | ·003436 | ·002894 | ·002417 | ·002002 | 1·9 |
| 2·0 | ·005752 | ·005046 | ·004393 | ·003794 | ·003251 | ·002763 | ·002328 | ·001946 | ·001613 | 2·0 |
| 2·1 | ·004568 | ·004010 | ·003493 | ·003019 | ·002589 | ·002202 | ·001857 | ·001553 | ·001288 | 2·1 |
| 2·2 | ·003595 | ·003158 | ·002753 | ·002381 | ·002043 | ·001739 | ·001468 | ·001228 | ·001020 | 2·2 |
| 2·3 | ·002804 | ·002465 | ·002151 | ·001862 | ·001598 | ·001361 | ·001150 | ·000963 | ·000800 | 2·3 |
| 2·4 | ·002168 | ·001907 | ·001665 | ·001442 | ·001239 | ·001056 | ·000893 | ·000748 | ·000622 | 2·4 |
| 2·5 | ·001661 | ·001462 | ·001277 | ·001107 | ·000952 | ·000812 | ·000687 | ·000576 | ·000479 | 2·5 |
| 2·6 | ·001261 | ·001111 | ·000971 | ·000842 | ·000725 | ·000619 | ·000524 | ·000440 | ·000366 | 2·6 |

| $k$ | $r=\cdot10$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·021103 | ·016965 | ·013518 | ·010677 | ·008358 | ·006484 | ·004985 | ·003798 | ·002867 | 0·0 |
| 0·1 | ·019685 | ·015833 | ·012623 | ·009975 | ·007813 | ·006064 | ·004665 | ·003556 | ·002686 | 0·1 |
| 0·2 | ·018249 | ·014686 | ·011715 | ·009263 | ·007259 | ·005638 | ·004339 | ·003309 | ·002501 | 0·2 |
| 0·3 | ·016811 | ·013536 | ·010804 | ·008547 | ·006702 | ·005208 | ·004011 | ·003061 | ·002315 | 0·3 |
| 0·4 | ·015384 | ·012394 | ·009898 | ·007836 | ·006147 | ·004780 | ·003683 | ·002813 | ·002128 | 0·4 |
| 0·5 | ·013983 | ·011272 | ·009008 | ·007135 | ·005601 | ·004358 | ·003360 | ·002567 | ·001944 | 0·5 |
| 0·6 | ·012621 | ·010180 | ·008140 | ·006452 | ·005068 | ·003946 | ·003044 | ·002327 | ·001763 | 0·6 |
| 0·7 | ·011310 | ·009129 | ·007304 | ·005793 | ·004553 | ·003547 | ·002738 | ·002095 | ·001588 | 0·7 |
| 0·8 | ·010061 | ·008126 | ·006506 | ·005163 | ·004061 | ·003166 | ·002446 | ·001872 | ·001420 | 0·8 |
| 0·9 | ·008883 | ·007179 | ·005752 | ·004568 | ·003595 | ·002804 | ·002168 | ·001661 | ·001261 | 0·9 |
| 1·0 | ·007783 | ·006294 | ·005046 | ·004010 | ·003158 | ·002465 | ·001907 | ·001462 | ·001111 | 1·0 |
| 1·1 | ·006766 | ·005476 | ·004393 | ·003493 | ·002753 | ·002151 | ·001665 | ·001277 | ·000971 | 1·1 |
| 1·2 | ·005836 | ·004726 | ·003794 | ·003019 | ·002381 | ·001862 | ·001442 | ·001107 | ·000842 | 1·2 |
| 1·3 | ·004993 | ·004046 | ·003251 | ·002589 | ·002043 | ·001598 | ·001239 | ·000952 | ·000725 | 1·3 |
| 1·4 | ·004237 | ·003436 | ·002763 | ·002202 | ·001739 | ·001361 | ·001056 | ·000812 | ·000619 | 1·4 |
| 1·5 | ·003566 | ·002894 | ·002328 | ·001857 | ·001468 | ·001150 | ·000893 | ·000687 | ·000524 | 1·5 |
| 1·6 | ·002976 | ·002417 | ·001946 | ·001553 | ·001228 | ·000963 | ·000748 | ·000576 | ·000440 | 1·6 |
| 1·7 | ·002463 | ·002002 | ·001613 | ·001288 | ·001020 | ·000800 | ·000622 | ·000479 | ·000366 | 1·7 |
| 1·8 | ·002020 | ·001643 | ·001325 | ·001059 | ·000839 | ·000659 | ·000513 | ·000395 | ·000302 | 1·8 |
| 1·9 | ·001643 | ·001338 | ·001080 | ·000864 | ·000685 | ·000538 | ·000419 | ·000323 | ·000247 | 1·9 |
| 2·0 | ·001325 | ·001080 | ·000872 | ·000698 | ·000554 | ·000435 | ·000339 | ·000262 | ·000201 | 2·0 |
| 2·1 | ·001059 | ·000864 | ·000698 | ·000559 | ·000444 | ·000349 | ·000273 | ·000211 | ·000161 | 2·1 |
| 2·2 | ·000839 | ·000685 | ·000554 | ·000444 | ·000353 | ·000278 | ·000217 | ·000168 | ·000129 | 2·2 |
| 2·3 | ·000659 | ·000538 | ·000435 | ·000349 | ·000278 | ·000219 | ·000171 | ·000132 | ·000102 | 2·3 |
| 2·4 | ·000513 | ·000419 | ·000339 | ·000273 | ·000217 | ·000171 | ·000134 | ·000104 | ·000080 | 2·4 |
| 2·5 | ·000395 | ·000323 | ·000262 | ·000211 | ·000168 | ·000132 | ·000104 | ·000080 | ·000062 | 2·5 |
| 2·6 | ·000302 | ·000247 | ·000201 | ·000161 | ·000129 | ·000102 | ·000080 | ·000062 | ·000047 | 2·6 |

| $k$ | $r=\cdot15$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·273964 | ·253929 | ·233856 | ·213946 | ·194397 | ·175397 | ·157115 | ·139703 | ·123287 | 0·0 |
| 0·1 | ·253929 | ·235500 | ·217016 | ·198665 | ·180629 | ·163082 | ·146183 | ·130072 | ·114867 | 0·1 |
| 0·2 | ·233856 | ·217016 | ·200109 | ·183307 | ·166777 | ·150679 | ·135159 | ·120348 | ·106357 | 0·2 |
| 0·3 | ·213946 | ·198665 | ·183307 | ·168028 | ·152981 | ·138312 | ·124155 | ·110631 | ·097843 | 0·3 |
| 0·4 | ·194397 | ·180629 | ·166777 | ·152981 | ·139380 | ·126106 | ·113283 | ·101019 | ·089411 | 0·4 |
| 0·5 | ·175397 | ·163082 | ·150679 | ·138312 | ·126106 | ·114181 | ·102648 | ·091607 | ·081144 | 0·5 |
| 0·6 | ·157115 | ·146183 | ·135159 | ·124155 | ·113283 | ·102648 | ·092352 | ·082484 | ·073122 | 0·6 |
| 0·7 | ·139703 | ·130072 | ·120348 | ·110631 | ·101019 | ·091607 | ·082484 | ·073730 | ·065416 | 0·7 |
| 0·8 | ·123287 | ·114867 | ·106357 | ·097843 | ·089411 | ·081144 | ·073122 | ·065416 | ·058088 | 0·8 |
| 0·9 | ·107964 | ·100662 | ·093273 | ·085871 | ·078532 | ·071328 | ·064330 | ·057598 | ·051190 | 0·9 |
| 1·0 | ·093807 | ·087525 | ·081161 | ·074778 | ·068441 | ·062214 | ·056156 | ·050323 | ·044763 | 1·0 |
| 1·1 | ·080859 | ·075499 | ·070061 | ·064602 | ·059175 | ·053835 | ·048634 | ·043620 | ·038835 | 1·1 |
| 1·2 | ·069136 | ·064600 | ·059993 | ·055361 | ·050752 | ·046211 | ·041782 | ·037508 | ·033423 | 1·2 |
| 1·3 | ·058628 | ·054822 | ·050952 | ·047055 | ·043173 | ·039343 | ·035604 | ·031989 | ·028532 | 1·3 |
| 1·4 | ·049305 | ·046139 | ·042915 | ·039665 | ·036422 | ·033219 | ·030088 | ·027058 | ·024156 | 1·4 |
| 1·5 | ·041117 | ·038505 | ·035842 | ·033155 | ·030470 | ·027814 | ·025215 | ·022696 | ·020280 | 1·5 |
| 1·6 | ·033998 | ·031862 | ·029682 | ·027479 | ·025274 | ·023092 | ·020952 | ·018876 | ·016883 | 1·6 |
| 1·7 | ·027870 | ·026139 | ·024370 | ·022579 | ·020785 | ·019007 | ·017262 | ·015566 | ·013935 | 1·7 |
| 1·8 | ·022650 | ·021259 | ·019835 | ·018393 | ·016946 | ·015510 | ·014098 | ·012725 | ·011403 | 1·8 |
| 1·9 | ·018246 | ·017139 | ·016004 | ·014852 | ·013696 | ·012546 | ·011414 | ·010312 | ·009249 | 1·9 |
| 2·0 | ·014569 | ·013696 | ·012799 | ·011888 | ·010971 | ·010059 | ·009160 | ·008283 | ·007437 | 2·0 |
| 2·1 | ·011530 | ·010847 | ·010145 | ·009431 | ·008711 | ·007994 | ·007286 | ·006595 | ·005926 | 2·1 |
| 2·2 | ·009044 | ·008515 | ·007970 | ·007415 | ·006855 | ·006296 | ·005744 | ·005204 | ·004681 | 2·2 |
| 2·3 | ·007030 | ·006623 | ·006205 | ·005777 | ·005346 | ·004914 | ·004487 | ·004069 | ·003664 | 2·3 |
| 2·4 | ·005415 | ·005106 | ·004787 | ·004461 | ·004131 | ·003801 | ·003474 | ·003153 | ·002842 | 2·4 |
| 2·5 | ·004133 | ·003900 | ·003659 | ·003413 | ·003163 | ·002913 | ·002665 | ·002421 | ·002184 | 2·5 |
| 2·6 | ·003126 | ·002952 | ·002772 | ·002587 | ·002400 | ·002213 | ·002026 | ·001842 | ·001664 | 2·6 |

| $k$ | $r=\cdot15$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·107964 | ·093807 | ·080859 | ·069136 | ·058628 | ·049305 | ·041117 | ·033998 | ·027870 | 0·0 |
| 0·1 | ·100662 | ·087525 | ·075499 | ·064600 | ·054822 | ·046139 | ·038505 | ·031862 | ·026139 | 0·1 |
| 0·2 | ·093273 | ·081161 | ·070061 | ·059993 | ·050952 | ·042915 | ·035842 | ·029682 | ·024370 | 0·2 |
| 0·3 | ·085871 | ·074778 | ·064602 | ·055361 | ·047055 | ·039665 | ·033155 | ·027479 | ·022579 | 0·3 |
| 0·4 | ·078532 | ·068441 | ·059175 | ·050752 | ·043173 | ·036422 | ·030470 | ·025273 | ·020785 | 0·4 |
| 0·5 | ·071328 | ·062214 | ·053835 | ·046211 | ·039343 | ·033219 | ·027814 | ·023092 | ·019007 | 0·5 |
| 0·6 | ·064330 | ·056156 | ·048634 | ·041782 | ·035604 | ·030088 | ·025215 | ·020952 | ·017262 | 0·6 |
| 0·7 | ·057598 | ·050323 | ·043620 | ·037508 | ·031989 | ·027058 | ·022696 | ·018876 | ·015566 | 0·7 |
| 0·8 | ·051190 | ·044763 | ·038835 | ·033423 | ·028532 | ·024156 | ·020280 | ·016883 | ·013935 | 0·8 |
| 0·9 | ·045151 | ·039517 | ·034315 | ·029560 | ·025257 | ·021403 | ·017987 | ·014988 | ·012382 | 0·9 |
| 1·0 | ·039517 | ·034618 | ·030088 | ·025943 | ·022188 | ·018820 | ·015831 | ·013204 | ·010920 | 1·0 |
| 1·1 | ·034315 | ·030088 | ·026176 | ·022591 | ·019339 | ·016420 | ·013825 | ·011543 | ·009555 | 1·1 |
| 1·2 | ·029560 | ·025943 | ·022591 | ·019515 | ·016723 | ·014212 | ·011979 | ·010011 | ·008296 | 1·2 |
| 1·3 | ·025257 | ·022188 | ·019339 | ·016723 | ·014344 | ·012203 | ·010295 | ·008613 | ·007144 | 1·3 |
| 1·4 | ·021403 | ·018820 | ·016420 | ·014212 | ·012203 | ·010392 | ·008776 | ·007350 | ·006103 | 1·4 |
| 1·5 | ·017987 | ·015831 | ·013825 | ·011979 | ·010295 | ·008776 | ·007420 | ·006220 | ·005170 | 1·5 |
| 1·6 | ·014988 | ·013204 | ·011543 | ·010011 | ·008613 | ·007350 | ·006220 | ·005220 | ·004344 | 1·6 |
| 1·7 | ·012382 | ·010920 | ·009555 | ·008296 | ·007144 | ·006103 | ·005170 | ·004344 | ·003618 | 1·7 |
| 1·8 | ·010142 | ·008953 | ·007842 | ·006815 | ·005875 | ·005024 | ·004261 | ·003584 | ·002988 | 1·8 |
| 1·9 | ·008235 | ·007276 | ·006380 | ·005550 | ·004790 | ·004100 | ·003481 | ·002931 | ·002447 | 1·9 |
| 2·0 | ·006627 | ·005862 | ·005145 | ·004481 | ·003871 | ·003317 | ·002819 | ·002376 | ·001986 | 2·0 |
| 2·1 | ·005287 | ·004681 | ·004113 | ·003585 | ·003101 | ·002660 | ·002263 | ·001910 | ·001598 | 2·1 |
| 2·2 | ·004180 | ·003704 | ·003258 | ·002843 | ·002462 | ·002114 | ·001801 | ·001521 | ·001274 | 2·2 |
| 2·3 | ·003275 | ·002905 | ·002558 | ·002235 | ·001937 | ·001665 | ·001420 | ·001201 | ·001007 | 2·3 |
| 2·4 | ·002543 | ·002258 | ·001990 | ·001740 | ·001510 | ·001300 | ·001109 | ·000939 | ·000789 | 2·4 |
| 2·5 | ·001956 | ·001739 | ·001534 | ·001343 | ·001167 | ·001005 | ·000859 | ·000728 | ·000612 | 2·5 |
| 2·6 | ·001492 | ·001327 | ·001172 | ·001027 | ·000893 | ·000770 | ·000659 | ·000559 | ·000471 | 2·6 |

| $k$ | $r=\cdot15$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·022650 | ·018246 | ·014569 | ·011530 | ·009044 | ·007030 | ·005415 | ·004133 | ·003126 | 0·0 |
| 0·1 | ·021259 | ·017139 | ·013696 | ·010847 | ·008515 | ·006623 | ·005106 | ·003900 | ·002952 | 0·1 |
| 0·2 | ·019835 | ·016004 | ·012799 | ·010145 | ·007970 | ·006205 | ·004787 | ·003659 | ·002772 | 0·2 |
| 0·3 | ·018393 | ·014852 | ·011888 | ·009431 | ·007415 | ·005777 | ·004461 | ·003413 | ·002587 | 0·3 |
| 0·4 | ·016946 | ·013696 | ·010971 | ·008711 | ·006855 | ·005346 | ·004131 | ·003163 | ·002400 | 0·4 |
| 0·5 | ·015510 | ·012546 | ·010059 | ·007994 | ·006296 | ·004914 | ·003801 | ·002913 | ·002213 | 0·5 |
| 0·6 | ·014098 | ·011414 | ·009160 | ·007286 | ·005744 | ·004487 | ·003474 | ·002665 | ·002026 | 0·6 |
| 0·7 | ·012725 | ·010312 | ·008283 | ·006595 | ·005204 | ·004069 | ·003153 | ·002421 | ·001842 | 0·7 |
| 0·8 | ·011403 | ·009249 | ·007437 | ·005926 | ·004681 | ·003664 | ·002842 | ·002184 | ·001664 | 0·8 |
| 0·9 | ·010142 | ·008235 | ·006627 | ·005287 | ·004180 | ·003275 | ·002543 | ·001956 | ·001492 | 0·9 |
| 1·0 | ·008953 | ·007276 | ·005862 | ·004681 | ·003704 | ·002905 | ·002258 | ·001739 | ·001327 | 1·0 |
| 1·1 | ·007842 | ·006380 | ·005145 | ·004113 | ·003258 | ·002558 | ·001990 | ·001534 | ·001172 | 1·1 |
| 1·2 | ·006815 | ·005550 | ·004481 | ·003585 | ·002843 | ·002235 | ·001740 | ·001343 | ·001027 | 1·2 |
| 1·3 | ·005875 | ·004790 | ·003871 | ·003101 | ·002462 | ·001937 | ·001510 | ·001167 | ·000893 | 1·3 |
| 1·4 | ·005024 | ·004100 | ·003317 | ·002660 | ·002114 | ·001665 | ·001300 | ·001005 | ·000770 | 1·4 |
| 1·5 | ·004261 | ·003481 | ·002819 | ·002263 | ·001801 | ·001420 | ·001109 | ·000859 | ·000659 | 1·5 |
| 1·6 | ·003584 | ·002931 | ·002376 | ·001910 | ·001521 | ·001201 | ·000939 | ·000728 | ·000559 | 1·6 |
| 1·7 | ·002988 | ·002447 | ·001986 | ·001598 | ·001274 | ·001007 | ·000789 | ·000612 | ·000471 | 1·7 |
| 1·8 | ·002471 | ·002025 | ·001646 | ·001325 | ·001058 | ·000837 | ·000656 | ·000510 | ·000393 | 1·8 |
| 1·9 | ·002025 | ·001662 | ·001352 | ·001090 | ·000871 | ·000690 | ·000542 | ·000421 | ·000325 | 1·9 |
| 2·0 | ·001646 | ·001352 | ·001101 | ·000889 | ·000711 | ·000564 | ·000443 | ·000345 | ·000266 | 2·0 |
| 2·1 | ·001325 | ·001090 | ·000889 | ·000718 | ·000575 | ·000457 | ·000359 | ·000280 | ·000217 | 2·1 |
| 2·2 | ·001058 | ·000871 | ·000711 | ·000575 | ·000461 | ·000367 | ·000289 | ·000226 | ·000174 | 2·2 |
| 2·3 | ·000837 | ·000690 | ·000564 | ·000457 | ·000367 | ·000292 | ·000230 | ·000180 | ·000139 | 2·3 |
| 2·4 | ·000656 | ·000542 | ·000443 | ·000359 | ·000289 | ·000230 | ·000182 | ·000142 | ·000110 | 2·4 |
| 2·5 | ·000510 | ·000421 | ·000345 | ·000280 | ·000226 | ·000180 | ·000142 | ·000111 | ·000086 | 2·5 |
| 2·6 | ·000393 | ·000325 | ·000266 | ·000217 | ·000174 | ·000139 | ·000110 | ·000086 | ·000067 | 2·6 |

| $k$ | $r=\cdot20$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·282047 | ·261971 | ·241774 | ·221662 | ·201840 | ·182502 | ·163829 | ·145981 | ·129097 | $0\cdot0$ |
| $0\cdot1$ | ·261971 | ·243515 | ·224922 | ·206384 | ·188087 | ·170215 | ·152934 | ·136397 | ·120731 | $0\cdot1$ |
| $0\cdot2$ | ·241774 | ·224922 | ·207922 | ·190949 | ·174174 | ·157765 | ·141879 | ·126655 | ·112215 | $0\cdot2$ |
| $0\cdot3$ | ·221662 | ·206384 | ·190949 | ·175516 | ·160242 | ·145281 | ·130776 | ·116856 | ·103635 | $0\cdot3$ |
| $0\cdot4$ | ·201840 | ·188087 | ·174174 | ·160242 | ·146434 | ·132889 | ·119738 | ·107100 | ·095079 | $0\cdot4$ |
| $0\cdot5$ | ·182502 | ·170215 | ·157765 | ·145281 | ·132889 | ·120715 | ·108878 | ·097486 | ·086634 | $0\cdot5$ |
| $0\cdot6$ | ·163829 | ·152934 | ·141879 | ·130776 | ·119738 | ·108878 | ·098302 | ·088109 | ·078384 | $0\cdot6$ |
| $0\cdot7$ | ·145981 | ·136397 | ·126655 | ·116856 | ·107100 | ·097486 | ·088109 | ·079057 | ·070408 | $0\cdot7$ |
| $0\cdot8$ | ·129097 | ·120731 | ·112215 | ·103635 | ·095079 | ·086634 | ·078384 | ·070408 | ·062776 | $0\cdot8$ |
| $0\cdot9$ | ·113286 | ·106044 | ·098659 | ·091206 | ·083762 | ·076403 | ·069203 | ·062230 | ·055547 | $0\cdot9$ |
| $1\cdot0$ | ·098633 | ·092413 | ·086062 | ·079641 | ·073217 | ·066856 | ·060622 | ·054576 | ·048771 | $1\cdot0$ |
| $1\cdot1$ | ·085189 | ·079893 | ·074475 | ·068989 | ·063491 | ·058038 | ·052686 | ·047485 | ·042484 | $1\cdot1$ |
| $1\cdot2$ | ·072981 | ·068509 | ·063927 | ·059279 | ·054613 | ·049978 | ·045420 | ·040984 | ·036711 | $1\cdot2$ |
| $1\cdot3$ | ·062008 | ·058265 | ·054422 | ·050518 | ·046592 | ·042684 | ·038836 | ·035084 | ·031464 | $1\cdot3$ |
| $1\cdot4$ | ·052246 | ·049139 | ·045945 | ·042693 | ·039418 | ·036153 | ·032931 | ·029784 | ·026744 | $1\cdot4$ |
| $1\cdot5$ | ·043649 | ·041094 | ·038461 | ·035777 | ·033068 | ·030363 | ·027689 | ·025073 | ·022541 | $1\cdot5$ |
| $1\cdot6$ | ·036156 | ·034072 | ·031922 | ·029725 | ·027505 | ·025283 | ·023084 | ·020928 | ·018838 | $1\cdot6$ |
| $1\cdot7$ | ·029690 | ·028007 | ·026266 | ·024484 | ·022680 | ·020872 | ·019079 | ·017318 | ·015608 | $1\cdot7$ |
| $1\cdot8$ | ·024169 | ·022820 | ·021424 | ·019992 | ·018539 | ·017081 | ·015632 | ·014206 | ·012819 | $1\cdot8$ |
| $1\cdot9$ | ·019501 | ·018431 | ·017321 | ·016181 | ·015022 | ·013856 | ·012695 | ·011552 | ·010437 | $1\cdot9$ |
| $2\cdot0$ | ·015596 | ·014755 | ·013880 | ·012980 | ·012063 | ·011140 | ·010219 | ·009310 | ·008422 | $2\cdot0$ |
| $2\cdot1$ | ·012361 | ·011706 | ·011023 | ·010319 | ·009601 | ·008877 | ·008152 | ·007436 | ·006735 | $2\cdot1$ |
| $2\cdot2$ | ·009709 | ·009204 | ·008676 | ·008130 | ·007573 | ·007009 | ·006445 | ·005886 | ·005338 | $2\cdot2$ |
| $2\cdot3$ | ·007558 | ·007171 | ·006766 | ·006348 | ·005919 | ·005485 | ·005050 | ·004617 | ·004193 | $2\cdot3$ |
| $2\cdot4$ | ·005829 | ·005536 | ·005229 | ·004911 | ·004584 | ·004253 | ·003920 | ·003589 | ·003263 | $2\cdot4$ |
| $2\cdot5$ | ·004455 | ·004235 | ·004004 | ·003764 | ·003518 | ·003267 | ·003015 | ·002764 | ·002516 | $2\cdot5$ |
| $2\cdot6$ | ·003373 | ·003210 | ·003038 | ·002859 | ·002675 | ·002487 | ·002298 | ·002109 | ·001922 | $2\cdot6$ |

| $k$ | $r=\cdot20$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·113286 | ·098633 | ·085189 | ·072981 | ·062008 | ·052246 | ·043649 | ·036156 | ·029690 | $0\cdot0$ |
| $0\cdot1$ | ·106044 | ·092413 | ·079893 | ·068509 | ·058265 | ·049139 | ·041094 | ·034072 | ·028007 | $0\cdot1$ |
| $0\cdot2$ | ·098659 | ·086062 | ·074475 | ·063927 | ·054422 | ·045945 | ·038461 | ·031922 | ·026266 | $0\cdot2$ |
| $0\cdot3$ | ·091206 | ·079641 | ·068989 | ·059279 | ·050518 | ·042693 | ·035777 | ·029725 | ·024484 | $0\cdot3$ |
| $0\cdot4$ | ·083762 | ·073217 | ·063491 | ·054613 | ·046592 | ·039418 | ·033068 | ·027505 | ·022680 | $0\cdot4$ |
| $0\cdot5$ | ·076403 | ·066856 | ·058038 | ·049978 | ·042684 | ·036153 | ·030363 | ·025283 | ·020872 | $0\cdot5$ |
| $0\cdot6$ | ·069203 | ·060622 | ·052686 | ·045420 | ·038836 | ·032931 | ·027689 | ·023084 | ·019079 | $0\cdot6$ |
| $0\cdot7$ | ·062230 | ·054576 | ·047485 | ·040984 | ·035084 | ·029784 | ·025073 | ·020928 | ·017318 | $0\cdot7$ |
| $0\cdot8$ | ·055547 | ·048771 | ·042484 | ·036711 | ·031464 | ·026744 | ·022541 | ·018838 | ·015608 | $0\cdot8$ |
| $0\cdot9$ | ·049208 | ·043256 | ·037725 | ·032638 | ·028007 | ·023835 | ·020114 | ·016831 | ·013962 | $0\cdot9$ |
| $1\cdot0$ | ·043256 | ·038069 | ·033241 | ·028794 | ·024740 | ·021081 | ·017813 | ·014924 | ·012397 | $1\cdot0$ |
| $1\cdot1$ | ·037725 | ·033241 | ·029062 | ·025205 | ·021684 | ·018500 | ·015653 | ·013132 | ·010922 | $1\cdot1$ |
| $1\cdot2$ | ·032638 | ·028794 | ·025205 | ·021889 | ·018855 | ·016108 | ·013646 | ·011463 | ·009548 | $1\cdot2$ |
| $1\cdot3$ | ·028007 | ·024740 | ·021684 | ·018855 | ·016262 | ·013911 | ·011801 | ·009927 | ·008279 | $1\cdot3$ |
| $1\cdot4$ | ·023835 | ·021081 | ·018500 | ·016108 | ·013911 | ·011916 | ·010123 | ·008527 | ·007121 | $1\cdot4$ |
| $1\cdot5$ | ·020114 | ·017813 | ·015653 | ·013646 | ·011801 | ·010123 | ·008611 | ·007263 | ·006074 | $1\cdot5$ |
| $1\cdot6$ | ·016831 | ·014924 | ·013132 | ·011463 | ·009927 | ·008527 | ·007263 | ·006135 | ·005138 | $1\cdot6$ |
| $1\cdot7$ | ·013962 | ·012397 | ·010922 | ·009548 | ·008279 | ·007121 | ·006074 | ·005138 | ·004309 | $1\cdot7$ |
| $1\cdot8$ | ·011483 | ·010208 | ·009006 | ·007883 | ·006845 | ·005896 | ·005036 | ·004266 | ·003583 | $1\cdot8$ |
| $1\cdot9$ | ·009361 | ·008333 | ·007361 | ·006452 | ·005610 | ·004839 | ·004140 | ·003512 | ·002954 | $1\cdot9$ |
| $2\cdot0$ | ·007563 | ·006742 | ·005963 | ·005234 | ·004558 | ·003937 | ·003372 | ·002865 | ·002413 | $2\cdot0$ |
| $2\cdot1$ | ·006056 | ·005406 | ·004788 | ·004209 | ·003670 | ·003174 | ·002723 | ·002317 | ·001955 | $2\cdot1$ |
| $2\cdot2$ | ·004806 | ·004296 | ·003810 | ·003353 | ·002928 | ·002536 | ·002179 | ·001857 | ·001569 | $2\cdot2$ |
| $2\cdot3$ | ·003780 | ·003383 | ·003004 | ·002648 | ·002316 | ·002009 | ·001728 | ·001475 | ·001248 | $2\cdot3$ |
| $2\cdot4$ | ·002946 | ·002639 | ·002348 | ·002072 | ·001814 | ·001576 | ·001358 | ·001161 | ·000984 | $2\cdot4$ |
| $2\cdot5$ | ·002274 | ·002041 | ·001818 | ·001606 | ·001409 | ·001225 | ·001058 | ·000905 | ·000768 | $2\cdot5$ |
| $2\cdot6$ | ·001740 | ·001563 | ·001394 | ·001234 | ·001084 | ·000944 | ·000816 | ·000699 | ·000594 | $2\cdot6$ |

| $k$ | $r=\cdot20$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·024169 | ·019501 | ·015596 | ·012361 | ·009709 | ·007558 | ·005829 | ·004455 | ·003373 | $0\cdot0$ |
| $0\cdot1$ | ·022820 | ·018431 | ·014755 | ·011706 | ·009204 | ·007171 | ·005536 | ·004235 | ·003210 | $0\cdot1$ |
| $0\cdot2$ | ·021424 | ·017321 | ·013880 | ·011023 | ·008676 | ·006766 | ·005229 | ·004004 | ·003038 | $0\cdot2$ |
| $0\cdot3$ | ·019992 | ·016181 | ·012980 | ·010319 | ·008130 | ·006348 | ·004911 | ·003764 | ·002859 | $0\cdot3$ |
| $0\cdot4$ | ·018539 | ·015022 | ·012063 | ·009601 | ·007573 | ·005919 | ·004584 | ·003518 | ·002675 | $0\cdot4$ |
| $0\cdot5$ | ·017081 | ·013856 | ·011140 | ·008877 | ·007009 | ·005485 | ·004253 | ·003267 | ·002487 | $0\cdot5$ |
| $0\cdot6$ | ·015632 | ·012695 | ·010219 | ·008152 | ·006445 | ·005050 | ·003920 | ·003015 | ·002298 | $0\cdot6$ |
| $0\cdot7$ | ·014206 | ·011552 | ·009310 | ·007436 | ·005886 | ·004617 | ·003589 | ·002764 | ·002109 | $0\cdot7$ |
| $0\cdot8$ | ·012819 | ·010437 | ·008422 | ·006735 | ·005338 | ·004193 | ·003263 | ·002516 | ·001922 | $0\cdot8$ |
| $0\cdot9$ | ·011483 | ·009361 | ·007563 | ·006056 | ·004806 | ·003780 | ·002946 | ·002274 | ·001740 | $0\cdot9$ |
| $1\cdot0$ | ·010208 | ·008333 | ·006742 | ·005406 | ·004296 | ·003383 | ·002639 | ·002041 | ·001563 | $1\cdot0$ |
| $1\cdot1$ | ·009006 | ·007361 | ·005963 | ·004788 | ·003810 | ·003004 | ·002348 | ·001818 | ·001394 | $1\cdot1$ |
| $1\cdot2$ | ·007883 | ·006452 | ·005234 | ·004209 | ·003353 | ·002648 | ·002072 | ·001606 | ·001234 | $1\cdot2$ |
| $1\cdot3$ | ·006845 | ·005610 | ·004558 | ·003670 | ·002928 | ·002316 | ·001814 | ·001409 | ·001084 | $1\cdot3$ |
| $1\cdot4$ | ·005896 | ·004839 | ·003937 | ·003174 | ·002536 | ·002009 | ·001576 | ·001225 | ·000944 | $1\cdot4$ |
| $1\cdot5$ | ·005036 | ·004140 | ·003372 | ·002723 | ·002179 | ·001728 | ·001358 | ·001058 | ·000816 | $1\cdot5$ |
| $1\cdot6$ | ·004266 | ·003512 | ·002865 | ·002317 | ·001857 | ·001475 | ·001161 | ·000905 | ·000699 | $1\cdot6$ |
| $1\cdot7$ | ·003583 | ·002954 | ·002413 | ·001955 | ·001569 | ·001248 | ·000984 | ·000768 | ·000594 | $1\cdot7$ |
| $1\cdot8$ | ·002984 | ·002463 | ·002016 | ·001635 | ·001314 | ·001047 | ·000826 | ·000646 | ·000501 | $1\cdot8$ |
| $1\cdot9$ | ·002463 | ·002037 | ·001669 | ·001356 | ·001091 | ·000871 | ·000688 | ·000539 | ·000419 | $1\cdot9$ |
| $2\cdot0$ | ·002016 | ·001669 | ·001370 | ·001114 | ·000898 | ·000718 | ·000569 | ·000446 | ·000347 | $2\cdot0$ |
| $2\cdot1$ | ·001635 | ·001356 | ·001114 | ·000908 | ·000733 | ·000587 | ·000465 | ·000366 | ·000285 | $2\cdot1$ |
| $2\cdot2$ | ·001314 | ·001091 | ·000898 | ·000733 | ·000593 | ·000475 | ·000378 | ·000297 | ·000232 | $2\cdot2$ |
| $2\cdot3$ | ·001047 | ·000871 | ·000718 | ·000587 | ·000475 | ·000382 | ·000304 | ·000239 | ·000187 | $2\cdot3$ |
| $2\cdot4$ | ·000826 | ·000688 | ·000569 | ·000465 | ·000378 | ·000304 | ·000242 | ·000191 | ·000149 | $2\cdot4$ |
| $2\cdot5$ | ·000646 | ·000539 | ·000446 | ·000366 | ·000297 | ·000239 | ·000191 | ·000151 | ·000118 | $2\cdot5$ |
| $2\cdot6$ | ·000501 | ·000419 | ·000347 | ·000285 | ·000232 | ·000187 | ·000149 | ·000118 | ·000093 | $2\cdot6$ |

| $k$ | $r=\cdot25$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·290215 | ·270096 | ·249772 | ·229452 | ·209348 | ·189662 | ·170586 | ·152291 | ·134927 | $0\cdot0$ |
| $0\cdot1$ | ·270096 | ·251617 | ·232916 | ·214188 | ·195627 | ·177422 | ·159752 | ·142778 | ·126642 | $0\cdot1$ |
| $0\cdot2$ | ·249772 | ·232916 | ·215828 | ·198685 | ·181666 | ·164945 | ·148686 | ·133042 | ·118145 | $0\cdot2$ |
| $0\cdot3$ | ·229452 | ·214188 | ·198685 | ·183105 | ·167609 | ·152357 | ·137501 | ·123181 | ·109521 | $0\cdot3$ |
| $0\cdot4$ | ·209348 | ·195627 | ·181666 | ·167609 | ·153602 | ·139790 | ·126313 | ·113298 | ·100861 | $0\cdot4$ |
| $0\cdot5$ | ·189662 | ·177422 | ·164945 | ·152357 | ·139790 | ·127375 | ·115238 | ·103496 | ·092254 | $0\cdot5$ |
| $0\cdot6$ | ·170586 | ·159752 | ·148686 | ·137501 | ·126313 | ·115238 | ·104390 | ·093875 | ·083789 | $0\cdot6$ |
| $0\cdot7$ | ·152291 | ·142778 | ·133042 | ·123181 | ·113298 | ·103496 | ·093875 | ·084532 | ·075552 | $0\cdot7$ |
| $0\cdot8$ | ·134927 | ·126642 | ·118145 | ·109521 | ·100861 | ·092254 | ·083789 | ·075552 | ·067620 | $0\cdot8$ |
| $0\cdot9$ | ·118617 | ·111461 | ·104106 | ·096626 | ·089099 | ·081603 | ·074216 | ·067012 | ·060061 | $0\cdot9$ |
| $1\cdot0$ | ·103456 | ·097326 | ·091014 | ·084580 | ·078091 | ·071617 | ·065223 | ·058975 | ·052933 | $1\cdot0$ |
| $1\cdot1$ | ·089507 | ·084302 | ·078930 | ·073442 | ·067897 | ·062351 | ·056863 | ·051489 | ·046282 | $1\cdot1$ |
| $1\cdot2$ | ·076806 | ·072424 | ·067892 | ·063252 | ·058553 | ·053844 | ·049174 | ·044590 | ·040140 | $1\cdot2$ |
| $1\cdot3$ | ·065362 | ·061705 | ·057914 | ·054026 | ·050078 | ·046114 | ·042173 | ·038298 | ·034527 | $1\cdot3$ |
| $1\cdot4$ | ·055155 | ·052131 | ·048988 | ·045758 | ·042471 | ·039163 | ·035867 | ·032619 | ·029452 | $1\cdot4$ |
| $1\cdot5$ | ·046146 | ·043667 | ·041086 | ·038426 | ·035714 | ·032977 | ·030246 | ·027547 | ·024910 | $1\cdot5$ |
| $1\cdot6$ | ·038276 | ·036263 | ·034161 | ·031991 | ·029773 | ·027530 | ·025286 | ·023065 | ·020888 | $1\cdot6$ |
| $1\cdot7$ | ·031473 | ·029852 | ·028157 | ·026402 | ·024605 | ·022783 | ·020957 | ·019144 | ·017364 | $1\cdot7$ |
| $1\cdot8$ | ·025651 | ·024359 | ·023004 | ·021599 | ·020155 | ·018689 | ·017216 | ·015750 | ·014308 | $1\cdot8$ |
| $1\cdot9$ | ·020721 | ·019700 | ·018628 | ·017512 | ·016364 | ·015195 | ·014017 | ·012843 | ·011685 | $1\cdot9$ |
| $2\cdot0$ | ·016589 | ·015790 | ·014949 | ·014072 | ·013167 | ·012244 | ·011311 | ·010379 | ·009458 | $2\cdot0$ |
| $2\cdot1$ | ·013161 | ·012542 | ·011889 | ·011205 | ·010499 | ·009776 | ·009045 | ·008312 | ·007586 | $2\cdot1$ |
| $2\cdot2$ | ·010347 | ·009872 | ·009369 | ·008842 | ·008296 | ·007735 | ·007167 | ·006596 | ·006030 | $2\cdot2$ |
| $2\cdot3$ | ·008061 | ·007699 | ·007316 | ·006913 | ·006495 | ·006065 | ·005627 | ·005187 | ·004749 | $2\cdot3$ |
| $2\cdot4$ | ·006222 | ·005950 | ·005660 | ·005355 | ·005038 | ·004711 | ·004377 | ·004041 | ·003705 | $2\cdot4$ |
| $2\cdot5$ | ·004758 | ·004555 | ·004339 | ·004110 | ·003872 | ·003625 | ·003374 | ·003119 | ·002864 | $2\cdot5$ |
| $2\cdot6$ | ·003605 | ·003455 | ·003295 | ·003125 | ·002948 | ·002764 | ·002576 | ·002385 | ·002194 | $2\cdot6$ |

| $k$ | $r=\cdot25$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·118617 | ·103456 | ·089507 | ·076806 | ·065362 | ·055155 | ·046146 | ·038276 | ·031473 | $0\cdot0$ |
| $0\cdot1$ | ·111461 | ·097326 | ·084302 | ·072424 | ·061705 | ·052131 | ·043667 | ·036263 | ·029852 | $0\cdot1$ |
| $0\cdot2$ | ·104106 | ·091014 | ·078930 | ·067892 | ·057914 | ·048988 | ·041086 | ·034161 | ·028157 | $0\cdot2$ |
| $0\cdot3$ | ·096626 | ·084580 | ·073442 | ·063252 | ·054026 | ·045758 | ·038426 | ·031991 | ·026402 | $0\cdot3$ |
| $0\cdot4$ | ·089099 | ·078091 | ·067897 | ·058553 | ·050078 | ·042471 | ·035714 | ·029773 | ·024605 | $0\cdot4$ |
| $0\cdot5$ | ·081603 | ·071617 | ·062351 | ·053844 | ·046114 | ·039163 | ·032977 | ·027530 | ·022783 | $0\cdot5$ |
| $0\cdot6$ | ·074216 | ·065223 | ·056863 | ·049174 | ·042173 | ·035867 | ·030246 | ·025286 | ·020957 | $0\cdot6$ |
| $0\cdot7$ | ·067012 | ·058975 | ·051489 | ·044590 | ·038298 | ·032619 | ·027547 | ·023065 | ·019144 | $0\cdot7$ |
| $0\cdot8$ | ·060061 | ·052933 | ·046282 | ·040140 | ·034527 | ·029452 | ·024910 | ·020888 | ·017364 | $0\cdot8$ |
| $0\cdot9$ | ·053424 | ·047153 | ·041289 | ·035863 | ·030896 | ·026395 | ·022359 | ·018778 | ·015635 | $0\cdot9$ |
| $1\cdot0$ | ·047153 | ·041680 | ·036552 | ·031797 | ·027435 | ·023475 | ·019917 | ·016755 | ·013972 | $1\cdot0$ |
| $1\cdot1$ | ·041289 | ·036552 | ·032104 | ·027972 | ·024173 | ·020716 | ·017605 | ·014834 | ·012390 | $1\cdot1$ |
| $1\cdot2$ | ·035863 | ·031797 | ·027972 | ·024410 | ·021128 | ·018137 | ·015438 | ·013030 | ·010902 | $1\cdot2$ |
| $1\cdot3$ | ·030896 | ·027435 | ·024173 | ·021128 | ·018318 | ·015750 | ·013429 | ·011353 | ·009515 | $1\cdot3$ |
| $1\cdot4$ | ·026395 | ·023475 | ·020716 | ·018137 | ·015750 | ·013565 | ·011586 | ·009812 | ·008237 | $1\cdot4$ |
| $1\cdot5$ | ·022359 | ·019917 | ·017605 | ·015438 | ·013429 | ·011586 | ·009912 | ·008409 | ·007072 | $1\cdot5$ |
| $1\cdot6$ | ·018778 | ·016755 | ·014834 | ·013030 | ·011353 | ·009812 | ·008409 | ·007146 | ·006020 | $1\cdot6$ |
| $1\cdot7$ | ·015635 | ·013972 | ·012390 | ·010902 | ·009515 | ·008237 | ·007072 | ·006020 | ·005081 | $1\cdot7$ |
| $1\cdot8$ | ·012903 | ·011550 | ·010259 | ·009042 | ·007905 | ·006856 | ·005896 | ·005028 | ·004252 | $1\cdot8$ |
| $1\cdot9$ | ·010555 | ·009463 | ·008420 | ·007433 | ·006510 | ·005656 | ·004873 | ·004163 | ·003526 | $1\cdot9$ |
| $2\cdot0$ | ·008557 | ·007684 | ·006848 | ·006056 | ·005313 | ·004624 | ·003991 | ·003416 | ·002899 | $2\cdot0$ |
| $2\cdot1$ | ·006874 | ·006183 | ·005520 | ·004890 | ·004298 | ·003747 | ·003240 | ·002778 | ·002362 | $2\cdot1$ |
| $2\cdot2$ | ·005472 | ·004931 | ·004409 | ·003912 | ·003445 | ·003008 | ·002606 | ·002239 | ·001907 | $2\cdot2$ |
| $2\cdot3$ | ·004317 | ·003896 | ·003489 | ·003102 | ·002736 | ·002393 | ·002077 | ·001788 | ·001526 | $2\cdot3$ |
| $2\cdot4$ | ·003374 | ·003050 | ·002736 | ·002436 | ·002153 | ·001887 | ·001640 | ·001414 | ·001209 | $2\cdot4$ |
| $2\cdot5$ | ·002612 | ·002365 | ·002126 | ·001896 | ·001678 | ·001473 | ·001283 | ·001109 | ·000950 | $2\cdot5$ |
| $2\cdot6$ | ·002004 | ·001817 | ·001636 | ·001462 | ·001296 | ·001140 | ·000995 | ·000861 | ·000739 | $2\cdot6$ |

| $k$ | $r=\cdot25$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·025651 | ·020721 | ·016589 | ·013161 | ·010347 | ·008061 | ·006222 | ·004758 | ·003605 | $0\cdot0$ |
| $0\cdot1$ | ·024359 | ·019700 | ·015790 | ·012542 | ·009872 | ·007699 | ·005950 | ·004555 | ·003455 | $0\cdot1$ |
| $0\cdot2$ | ·023004 | ·018628 | ·014949 | ·011889 | ·009369 | ·007316 | ·005660 | ·004339 | ·003295 | $0\cdot2$ |
| $0\cdot3$ | ·021599 | ·017512 | ·014072 | ·011205 | ·008842 | ·006913 | ·005355 | ·004110 | ·003125 | $0\cdot3$ |
| $0\cdot4$ | ·020155 | ·016364 | ·013167 | ·010499 | ·008296 | ·006495 | ·005038 | ·003872 | ·002948 | $0\cdot4$ |
| $0\cdot5$ | ·018689 | ·015195 | ·012244 | ·009776 | ·007735 | ·006065 | ·004711 | ·003625 | ·002764 | $0\cdot5$ |
| $0\cdot6$ | ·017216 | ·014017 | ·011311 | ·009045 | ·007167 | ·005627 | ·004377 | ·003374 | ·002576 | $0\cdot6$ |
| $0\cdot7$ | ·015750 | ·012843 | ·010379 | ·008312 | ·006596 | ·005187 | ·004041 | ·003119 | ·002385 | $0\cdot7$ |
| $0\cdot8$ | ·014308 | ·011685 | ·009458 | ·007586 | ·006030 | ·004749 | ·003705 | ·002864 | ·002194 | $0\cdot8$ |
| $0\cdot9$ | ·012903 | ·010555 | ·008557 | ·006874 | ·005472 | ·004317 | ·003374 | ·002612 | ·002004 | $0\cdot9$ |
| $1\cdot0$ | ·011550 | ·009463 | ·007684 | ·006183 | ·004931 | ·003896 | ·003050 | ·002365 | ·001817 | $1\cdot0$ |
| $1\cdot1$ | ·010259 | ·008420 | ·006848 | ·005520 | ·004409 | ·003489 | ·002736 | ·002126 | ·001636 | $1\cdot1$ |
| $1\cdot2$ | ·009042 | ·007433 | ·006056 | ·004890 | ·003912 | ·003102 | ·002436 | ·001896 | ·001462 | $1\cdot2$ |
| $1\cdot3$ | ·007905 | ·006510 | ·005313 | ·004298 | ·003445 | ·002736 | ·002153 | ·001678 | ·001296 | $1\cdot3$ |
| $1\cdot4$ | ·006856 | ·005656 | ·004624 | ·003747 | ·003008 | ·002393 | ·001887 | ·001473 | ·001140 | $1\cdot4$ |
| $1\cdot5$ | ·005896 | ·004873 | ·003991 | ·003240 | ·002606 | ·002077 | ·001640 | ·001283 | ·000995 | $1\cdot5$ |
| $1\cdot6$ | ·005028 | ·004163 | ·003416 | ·002778 | ·002239 | ·001788 | ·001414 | ·001109 | ·000861 | $1\cdot6$ |
| $1\cdot7$ | ·004252 | ·003526 | ·002899 | ·002362 | ·001907 | ·001526 | ·001209 | ·000950 | ·000739 | $1\cdot7$ |
| $1\cdot8$ | ·003564 | ·002962 | ·002439 | ·001991 | ·001610 | ·001291 | ·001025 | ·000807 | ·000629 | $1\cdot8$ |
| $1\cdot9$ | ·002962 | ·002466 | ·002034 | ·001664 | ·001348 | ·001083 | ·000862 | ·000679 | ·000531 | $1\cdot9$ |
| $2\cdot0$ | ·002439 | ·002034 | ·001682 | ·001378 | ·001119 | ·000900 | ·000718 | ·000567 | ·000444 | $2\cdot0$ |
| $2\cdot1$ | ·001991 | ·001664 | ·001378 | ·001131 | ·000920 | ·000742 | ·000593 | ·000469 | ·000368 | $2\cdot1$ |
| $2\cdot2$ | ·001610 | ·001348 | ·001119 | ·000920 | ·000750 | ·000606 | ·000485 | ·000385 | ·000302 | $2\cdot2$ |
| $2\cdot3$ | ·001291 | ·001083 | ·000900 | ·000742 | ·000606 | ·000490 | ·000393 | ·000313 | ·000246 | $2\cdot3$ |
| $2\cdot4$ | ·001025 | ·000862 | ·000718 | ·000593 | ·000485 | ·000393 | ·000316 | ·000252 | ·000199 | $2\cdot4$ |
| $2\cdot5$ | ·000807 | ·000679 | ·000567 | ·000469 | ·000385 | ·000313 | ·000252 | ·000201 | ·000159 | $2\cdot5$ |
| $2\cdot6$ | ·000629 | ·000531 | ·000444 | ·000368 | ·000302 | ·000246 | ·000199 | ·000159 | ·000126 | $2\cdot6$ |

| $k$ | $r=\cdot30$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·298493 | ·278330 | ·257873 | ·237337 | ·216939 | ·196893 | ·177399 | ·158641 | ·140782 | 0·0 |
| 0·1 | ·278330 | ·259830 | ·241021 | ·222100 | ·203268 | ·184722 | ·166651 | ·149228 | ·132606 | 0·1 |
| 0·2 | ·257873 | ·241021 | ·223850 | ·206540 | ·189274 | ·172235 | ·155596 | ·139521 | ·124154 | 0·2 |
| 0·3 | ·237337 | ·222100 | ·206540 | ·190819 | ·175103 | ·159558 | ·144347 | ·129620 | ·115511 | 0·3 |
| 0·4 | ·216939 | ·203268 | ·189274 | ·175103 | ·160904 | ·146828 | ·133023 | ·119628 | ·106767 | 0·4 |
| 0·5 | ·196893 | ·184722 | ·172235 | ·159558 | ·146828 | ·134179 | ·121745 | ·109652 | ·098015 | 0·5 |
| 0·6 | ·177399 | ·166651 | ·155596 | ·144347 | ·133023 | ·121745 | ·110632 | ·099798 | ·089349 | 0·6 |
| 0·7 | ·158641 | ·149228 | ·139521 | ·129620 | ·119628 | ·109652 | ·099798 | ·090168 | ·080859 | 0·7 |
| 0·8 | ·140782 | ·132606 | ·124154 | ·115511 | ·106767 | ·098015 | ·089349 | ·080859 | ·072631 | 0·8 |
| 0·9 | ·123958 | ·116917 | ·109620 | ·102139 | ·094551 | ·086936 | ·079378 | ·071954 | ·064742 | 0·9 |
| 1·0 | ·108275 | ·102264 | ·096019 | ·089599 | ·083070 | ·076501 | ·069964 | ·063528 | ·057259 | 1·0 |
| 1·1 | ·093808 | ·088722 | ·083424 | ·077962 | ·072394 | ·066777 | ·061172 | ·055639 | ·050236 | 1·1 |
| 1·2 | ·080604 | ·076339 | ·071884 | ·067279 | ·062571 | ·057810 | ·053047 | ·048332 | ·043715 | 1·2 |
| 1·3 | ·068679 | ·065135 | ·061422 | ·057574 | ·053630 | ·049630 | ·045618 | ·041635 | ·037725 | 1·3 |
| 1·4 | ·058021 | ·055103 | ·052037 | ·048852 | ·045577 | ·042247 | ·038897 | ·035564 | ·032282 | 1·4 |
| 1·5 | ·048596 | ·046215 | ·043708 | ·041094 | ·038400 | ·035653 | ·032882 | ·030117 | ·027387 | 1·5 |
| 1·6 | ·040348 | ·038424 | ·036392 | ·034268 | ·032072 | ·029827 | ·027556 | ·025283 | ·023034 | 1·6 |
| 1·7 | ·033206 | ·031665 | ·030034 | ·028324 | ·026551 | ·024733 | ·022889 | ·021039 | ·019202 | 1·7 |
| 1·8 | ·027085 | ·025864 | ·024566 | ·023203 | ·021785 | ·020327 | ·018844 | ·017351 | ·015865 | 1·8 |
| 1·9 | ·021895 | ·020936 | ·019914 | ·018837 | ·017714 | ·016556 | ·015374 | ·014181 | ·012991 | 1·9 |
| 2·0 | ·017540 | ·016794 | ·015997 | ·015154 | ·014273 | ·013362 | ·012429 | ·011485 | ·010541 | 2·0 |
| 2·1 | ·013923 | ·013349 | ·012733 | ·012080 | ·011396 | ·010686 | ·009957 | ·009217 | ·008474 | 2·1 |
| 2·2 | ·010951 | ·010513 | ·010042 | ·009541 | ·009015 | ·008467 | ·007903 | ·007329 | ·006750 | 2·2 |
| 2·3 | ·008534 | ·008203 | ·007847 | ·007466 | ·007065 | ·006646 | ·006214 | ·005773 | ·005327 | 2·3 |
| 2·4 | ·006589 | ·006342 | ·006074 | ·005788 | ·005486 | ·005169 | ·004841 | ·004505 | ·004165 | 2·4 |
| 2·5 | ·005040 | ·004857 | ·004658 | ·004445 | ·004219 | ·003982 | ·003736 | ·003482 | ·003225 | 2·5 |
| 2·6 | ·003819 | ·003685 | ·003539 | ·003382 | ·003215 | ·003039 | ·002855 | ·002667 | ·002474 | 2·6 |

| $k$ | $r=\cdot30$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·123958 | ·108275 | ·093808 | ·080604 | ·068679 | ·058021 | ·048596 | ·040348 | ·033206 | 0·0 |
| 0·1 | ·116917 | ·102264 | ·088722 | ·076339 | ·065135 | ·055103 | ·046215 | ·038424 | ·031665 | 0·1 |
| 0·2 | ·109620 | ·096019 | ·083424 | ·071884 | ·061422 | ·052037 | ·043708 | ·036392 | ·030034 | 0·2 |
| 0·3 | ·102139 | ·089599 | ·077962 | ·067279 | ·057574 | ·048852 | ·041094 | ·034268 | ·028324 | 0·3 |
| 0·4 | ·094551 | ·083070 | ·072394 | ·062571 | ·053630 | ·045577 | ·038400 | ·032072 | ·026551 | 0·4 |
| 0·5 | ·086936 | ·076501 | ·066777 | ·057810 | ·049630 | ·042247 | ·035653 | ·029827 | ·024733 | 0·5 |
| 0·6 | ·079378 | ·069964 | ·061172 | ·053047 | ·045618 | ·038897 | ·032882 | ·027556 | ·022889 | 0·6 |
| 0·7 | ·071954 | ·063528 | ·055639 | ·048332 | ·041635 | ·035564 | ·030117 | ·025283 | ·021039 | 0·7 |
| 0·8 | ·064742 | ·057259 | ·050236 | ·043715 | ·037725 | ·032282 | ·027387 | ·023034 | ·019202 | 0·8 |
| 0·9 | ·057810 | ·051217 | ·045015 | ·039243 | ·033928 | ·029086 | ·024722 | ·020831 | ·017398 | 0·9 |
| 1·0 | ·051217 | ·045458 | ·040026 | ·034958 | ·030279 | ·026007 | ·022147 | ·018697 | ·015646 | 1·0 |
| 1·1 | ·045015 | ·040026 | ·035308 | ·030895 | ·026811 | ·023073 | ·019686 | ·016652 | ·013962 | 1·1 |
| 1·2 | ·039243 | ·034958 | ·030895 | ·027085 | ·023551 | ·020306 | ·017360 | ·014714 | ·012362 | 1·2 |
| 1·3 | ·033928 | ·030279 | ·026811 | ·023551 | ·020517 | ·017726 | ·015185 | ·012896 | ·010857 | 1·3 |
| 1·4 | ·029086 | ·026007 | ·023073 | ·020306 | ·017726 | ·015345 | ·013172 | ·011210 | ·009457 | 1·4 |
| 1·5 | ·024722 | ·022147 | ·019686 | ·017360 | ·015185 | ·013172 | ·011330 | ·009662 | ·008168 | 1·5 |
| 1·6 | ·020831 | ·018697 | ·016652 | ·014714 | ·012896 | ·011210 | ·009662 | ·008257 | ·006996 | 1·6 |
| 1·7 | ·017398 | ·015646 | ·013962 | ·012362 | ·010857 | ·009457 | ·008168 | ·006996 | ·005940 | 1·7 |
| 1·8 | ·014402 | ·012977 | ·011603 | ·010294 | ·009059 | ·007908 | ·006845 | ·005875 | ·004999 | 1·8 |
| 1·9 | ·011815 | ·010666 | ·009556 | ·008495 | ·007492 | ·006553 | ·005685 | ·004890 | ·004170 | 1·9 |
| 2·0 | ·009605 | ·008688 | ·007799 | ·006948 | ·006140 | ·005382 | ·004679 | ·004033 | ·003447 | 2·0 |
| 2·1 | ·007737 | ·007012 | ·006307 | ·005630 | ·004986 | ·004380 | ·003816 | ·003296 | ·002824 | 2·1 |
| 2·2 | ·006174 | ·005607 | ·005054 | ·004520 | ·004011 | ·003531 | ·003083 | ·002670 | ·002292 | 2·2 |
| 2·3 | ·004882 | ·004442 | ·004012 | ·003595 | ·003197 | ·002821 | ·002468 | ·002142 | ·001843 | 2·3 |
| 2·4 | ·003824 | ·003486 | ·003155 | ·002833 | ·002525 | ·002232 | ·001958 | ·001703 | ·001468 | 2·4 |
| 2·5 | ·002967 | ·002710 | ·002457 | ·002211 | ·001975 | ·001750 | ·001538 | ·001341 | ·001159 | 2·5 |
| 2·6 | ·002280 | ·002087 | ·001896 | ·001710 | ·001530 | ·001359 | ·001197 | ·001046 | ·000906 | 2·6 |

| $k$ | $r=\cdot30$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·027085 | ·021895 | ·017540 | ·013923 | ·010951 | ·008534 | ·006589 | ·005040 | ·003819 | $0\cdot0$ |
| $0\cdot1$ | ·025864 | ·020936 | ·016794 | ·013349 | ·010513 | ·008203 | ·006342 | ·004857 | ·003685 | $0\cdot1$ |
| $0\cdot2$ | ·024566 | ·019914 | ·015997 | ·012733 | ·010042 | ·007847 | ·006074 | ·004658 | ·003539 | $0\cdot2$ |
| $0\cdot3$ | ·023203 | ·018837 | ·015154 | ·012080 | ·009541 | ·007466 | ·005788 | ·004445 | ·003382 | $0\cdot3$ |
| $0\cdot4$ | ·021785 | ·017714 | ·014273 | ·011396 | ·009015 | ·007065 | ·005486 | ·004219 | ·003215 | $0\cdot4$ |
| $0\cdot5$ | ·020327 | ·016556 | ·013362 | ·010686 | ·008467 | ·006646 | ·005169 | ·003982 | ·003039 | $0\cdot5$ |
| $0\cdot6$ | ·018844 | ·015374 | ·012429 | ·009957 | ·007903 | ·006214 | ·004841 | ·003736 | ·002855 | $0\cdot6$ |
| $0\cdot7$ | ·017351 | ·014181 | ·011485 | ·009217 | ·007329 | ·005773 | ·004505 | ·003482 | ·002667 | $0\cdot7$ |
| $0\cdot8$ | ·015865 | ·012991 | ·010541 | ·008474 | ·006750 | ·005327 | ·004165 | ·003225 | ·002474 | $0\cdot8$ |
| $0\cdot9$ | ·014402 | ·011815 | ·009605 | ·007737 | ·006174 | ·004882 | ·003824 | ·002967 | ·002280 | $0\cdot9$ |
| $1\cdot0$ | ·012977 | ·010666 | ·008688 | ·007012 | ·005607 | ·004442 | ·003486 | ·002710 | ·002087 | $1\cdot0$ |
| $1\cdot1$ | ·011603 | ·009556 | ·007799 | ·006307 | ·005054 | ·004012 | ·003155 | ·002457 | ·001896 | $1\cdot1$ |
| $1\cdot2$ | ·010294 | ·008495 | ·006948 | ·005630 | ·004520 | ·003595 | ·002833 | ·002211 | ·001710 | $1\cdot2$ |
| $1\cdot3$ | ·009059 | ·007492 | ·006140 | ·004986 | ·004011 | ·003197 | ·002525 | ·001975 | ·001530 | $1\cdot3$ |
| $1\cdot4$ | ·007908 | ·006553 | ·005382 | ·004380 | ·003531 | ·002821 | ·002232 | ·001750 | ·001359 | $1\cdot4$ |
| $1\cdot5$ | ·006845 | ·005685 | ·004679 | ·003816 | ·003083 | ·002468 | ·001958 | ·001538 | ·001197 | $1\cdot5$ |
| $1\cdot6$ | ·005875 | ·004890 | ·004033 | ·003296 | ·002670 | ·002142 | ·001703 | ·001341 | ·001046 | $1\cdot6$ |
| $1\cdot7$ | ·004999 | ·004170 | ·003447 | ·002824 | ·002292 | ·001843 | ·001468 | ·001159 | ·000906 | $1\cdot7$ |
| $1\cdot8$ | ·004216 | ·003525 | ·002920 | ·002398 | ·001951 | ·001572 | ·001255 | ·000993 | ·000778 | $1\cdot8$ |
| $1\cdot9$ | ·003525 | ·002953 | ·002453 | ·002018 | ·001646 | ·001329 | ·001064 | ·000844 | ·000663 | $1\cdot9$ |
| $2\cdot0$ | ·002920 | ·002453 | ·002041 | ·001684 | ·001376 | ·001114 | ·000894 | ·000710 | ·000559 | $2\cdot0$ |
| $2\cdot1$ | ·002398 | ·002018 | ·001684 | ·001392 | ·001140 | ·000925 | ·000744 | ·000593 | ·000468 | $2\cdot1$ |
| $2\cdot2$ | ·001951 | ·001646 | ·001376 | ·001140 | ·000936 | ·000762 | ·000614 | ·000490 | ·000388 | $2\cdot2$ |
| $2\cdot3$ | ·001572 | ·001329 | ·001114 | ·000925 | ·000762 | ·000621 | ·000502 | ·000402 | ·000319 | $2\cdot3$ |
| $2\cdot4$ | ·001255 | ·001064 | ·000894 | ·000744 | ·000614 | ·000502 | ·000407 | ·000326 | ·000259 | $2\cdot4$ |
| $2\cdot5$ | ·000993 | ·000844 | ·000710 | ·000593 | ·000490 | ·000402 | ·000326 | ·000262 | ·000209 | $2\cdot5$ |
| $2\cdot6$ | ·000778 | ·000663 | ·000559 | ·000468 | ·000388 | ·000319 | ·000259 | ·000209 | ·000167 | $2\cdot6$ |

| $k$ | $r=\cdot35$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·306909 | ·286699 | ·266102 | ·245340 | ·224635 | ·204210 | ·184280 | ·165040 | ·146666 | $0\cdot0$ |
| $0\cdot1$ | ·286699 | ·268182 | ·249265 | ·230146 | ·211033 | ·192133 | ·173644 | ·155754 | ·138630 | $0\cdot1$ |
| $0\cdot2$ | ·266102 | ·249265 | ·232016 | ·214539 | ·197022 | ·179656 | ·162625 | ·146105 | ·130253 | $0\cdot2$ |
| $0\cdot3$ | ·245340 | ·230146 | ·214539 | ·198682 | ·182747 | ·166907 | ·151333 | ·136187 | ·121617 | $0\cdot3$ |
| $0\cdot4$ | ·224635 | ·211033 | ·197022 | ·182747 | ·168362 | ·154025 | ·139890 | ·126106 | ·112811 | $0\cdot4$ |
| $0\cdot5$ | ·204210 | ·192133 | ·179656 | ·166907 | ·154025 | ·141148 | ·128418 | ·115971 | ·103932 | $0\cdot5$ |
| $0\cdot6$ | ·184280 | ·173644 | ·162625 | ·151333 | ·139890 | ·128418 | ·117045 | ·105893 | ·095077 | $0\cdot6$ |
| $0\cdot7$ | ·165040 | ·155754 | ·146105 | ·136187 | ·126106 | ·115971 | ·105893 | ·095983 | ·086343 | $0\cdot7$ |
| $0\cdot8$ | ·146666 | ·138630 | ·130253 | ·121617 | ·112811 | ·103932 | ·095077 | ·086343 | ·077823 | $0\cdot8$ |
| $0\cdot9$ | ·129308 | ·122415 | ·115206 | ·107752 | ·100127 | ·092416 | ·084702 | ·077070 | ·069602 | $0\cdot9$ |
| $1\cdot0$ | ·113085 | ·107225 | ·101078 | ·094700 | ·088158 | ·081519 | ·074857 | ·068246 | ·061758 | $1\cdot0$ |
| $1\cdot1$ | ·098086 | ·093150 | ·087955 | ·082549 | ·076984 | ·071320 | ·065619 | ·059943 | ·054355 | $1\cdot1$ |
| $1\cdot2$ | ·084366 | ·080247 | ·075897 | ·071356 | ·066667 | ·061879 | ·057043 | ·052214 | ·047444 | $1\cdot2$ |
| $1\cdot3$ | ·071950 | ·068545 | ·064937 | ·061158 | ·057244 | ·053233 | ·049169 | ·045098 | ·041063 | $1\cdot3$ |
| $1\cdot4$ | ·060834 | ·058046 | ·055082 | ·051967 | ·048729 | ·045401 | ·042018 | ·038617 | ·035237 | $1\cdot4$ |
| $1\cdot5$ | ·050987 | ·048727 | ·046315 | ·043772 | ·041120 | ·038385 | ·035595 | ·032781 | ·029974 | $1\cdot5$ |
| $1\cdot6$ | ·042359 | ·040543 | ·038600 | ·036544 | ·034392 | ·032166 | ·029887 | ·027581 | ·025273 | $1\cdot6$ |
| $1\cdot7$ | ·034878 | ·033434 | ·031884 | ·030237 | ·028509 | ·026713 | ·024870 | ·022998 | ·021119 | $1\cdot7$ |
| $1\cdot8$ | ·028460 | ·027323 | ·026099 | ·024793 | ·023418 | ·021985 | ·020508 | ·019004 | ·017488 | $1\cdot8$ |
| $1\cdot9$ | ·023013 | ·022127 | ·021169 | ·020145 | ·019061 | ·017929 | ·016757 | ·015560 | ·014349 | $1\cdot9$ |
| $2\cdot0$ | ·018438 | ·017755 | ·017013 | ·016217 | ·015372 | ·014486 | ·013566 | ·012622 | ·011665 | $2\cdot0$ |
| $2\cdot1$ | ·014638 | ·014116 | ·013547 | ·012935 | ·012283 | ·011596 | ·010881 | ·010144 | ·009395 | $2\cdot1$ |
| $2\cdot2$ | ·011513 | ·011118 | ·010687 | ·010221 | ·009722 | ·009195 | ·008645 | ·008076 | ·007495 | $2\cdot2$ |
| $2\cdot3$ | ·008971 | ·008676 | ·008352 | ·008000 | ·007623 | ·007223 | ·006804 | ·006369 | ·005923 | $2\cdot3$ |
| $2\cdot4$ | ·006925 | ·006706 | ·006465 | ·006203 | ·005921 | ·005621 | ·005304 | ·004975 | ·004637 | $2\cdot4$ |
| $2\cdot5$ | ·005295 | ·005135 | ·004958 | ·004764 | ·004555 | ·004332 | ·004096 | ·003849 | ·003595 | $2\cdot5$ |
| $2\cdot6$ | ·004011 | ·003895 | ·003766 | ·003624 | ·003471 | ·003307 | ·003132 | ·002950 | ·002760 | $2\cdot6$ |

| $k$ | $r=\cdot35$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·129308 | ·113085 | ·098086 | ·084366 | ·071950 | ·060834 | ·050987 | ·042359 | ·034878 | 0·0 |
| 0·1 | ·122415 | ·107225 | ·093150 | ·080247 | ·068545 | ·058046 | ·048727 | ·040543 | ·033434 | 0·1 |
| 0·2 | ·115206 | ·101078 | ·087955 | ·075897 | ·064937 | ·055082 | ·046315 | ·038600 | ·031884 | 0·2 |
| 0·3 | ·107752 | ·094700 | ·082549 | ·071356 | ·061158 | ·051967 | ·043772 | ·036544 | ·030237 | 0·3 |
| 0·4 | ·100127 | ·088158 | ·076984 | ·066667 | ·057244 | ·048729 | ·041120 | ·034392 | ·028509 | 0·4 |
| 0·5 | ·092416 | ·081519 | ·071320 | ·061879 | ·053233 | ·045401 | ·038385 | ·032166 | ·026713 | 0·5 |
| 0·6 | ·084702 | ·074857 | ·065619 | ·057043 | ·049169 | ·042018 | ·035595 | ·029887 | ·024870 | 0·6 |
| 0·7 | ·077070 | ·068246 | ·059943 | ·052214 | ·045098 | ·038617 | ·032781 | ·027581 | ·022998 | 0·7 |
| 0·8 | ·069602 | ·061758 | ·054355 | ·047444 | ·041063 | ·035237 | ·029974 | ·025273 | ·021119 | 0·8 |
| 0·9 | ·062376 | ·055460 | ·048914 | ·042785 | ·037110 | ·031913 | ·027206 | ·022989 | ·019252 | 0·9 |
| 1·0 | ·055460 | ·049414 | ·043675 | ·038285 | ·033279 | ·028682 | ·024506 | ·020754 | ·017419 | 1·0 |
| 1·1 | ·048914 | ·043675 | ·038685 | ·033986 | ·029608 | ·025576 | ·021902 | ·018591 | ·015640 | 1·1 |
| 1·2 | ·042785 | ·038285 | ·033986 | ·029924 | ·026129 | ·022622 | ·019417 | ·016521 | ·013931 | 1·2 |
| 1·3 | ·037110 | ·033279 | ·029608 | ·026129 | ·022868 | ·019845 | ·017074 | ·014561 | ·012308 | 1·3 |
| 1·4 | ·031913 | ·028682 | ·025576 | ·022622 | ·019845 | ·017262 | ·014887 | ·012727 | ·010785 | 1·4 |
| 1·5 | ·027206 | ·024506 | ·021902 | ·019417 | ·017074 | ·014887 | ·012870 | ·011030 | ·009370 | 1·5 |
| 1·6 | ·022989 | ·020754 | ·018591 | ·016521 | ·014561 | ·012727 | ·011030 | ·009477 | ·008070 | 1·6 |
| 1·7 | ·019252 | ·017419 | ·015640 | ·013931 | ·012308 | ·010785 | ·009370 | ·008070 | ·006890 | 1·7 |
| 1·8 | ·015977 | ·014489 | ·013039 | ·011642 | ·010311 | ·009056 | ·007888 | ·006811 | ·005830 | 1·8 |
| 1·9 | ·013138 | ·011941 | ·010771 | ·009640 | ·008559 | ·007536 | ·006580 | ·005697 | ·004889 | 1·9 |
| 2·0 | ·010704 | ·009751 | ·008816 | ·007909 | ·007039 | ·006214 | ·005439 | ·004721 | ·004062 | 2·0 |
| 2·1 | ·008640 | ·007889 | ·007149 | ·006429 | ·005735 | ·005076 | ·004454 | ·003876 | ·003344 | 2·1 |
| 2·2 | ·006908 | ·006322 | ·005742 | ·005176 | ·004629 | ·004107 | ·003614 | ·003153 | ·002727 | 2·2 |
| 2·3 | ·005471 | ·005018 | ·004569 | ·004128 | ·003701 | ·003292 | ·002904 | ·002540 | ·002203 | 2·3 |
| 2·4 | ·004292 | ·003945 | ·003601 | ·003261 | ·002931 | ·002613 | ·002311 | ·002027 | ·001763 | 2·4 |
| 2·5 | ·003335 | ·003072 | ·002810 | ·002551 | ·002299 | ·002055 | ·001822 | ·001602 | ·001397 | 2·5 |
| 2·6 | ·002566 | ·002369 | ·002172 | ·001977 | ·001785 | ·001600 | ·001422 | ·001254 | ·001096 | 2·6 |

| $k$ | $r=\cdot35$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·028460 | ·023013 | ·018438 | ·014638 | ·011513 | ·008971 | ·006925 | ·005295 | ·004011 | 0·0 |
| 0·1 | ·027323 | ·022127 | ·017755 | ·014116 | ·011118 | ·008676 | ·006706 | ·005135 | ·003895 | 0·1 |
| 0·2 | ·026099 | ·021169 | ·017013 | ·013547 | ·010687 | ·008352 | ·006465 | ·004958 | ·003766 | 0·2 |
| 0·3 | ·024793 | ·020145 | ·016217 | ·012935 | ·010221 | ·008000 | ·006203 | ·004764 | ·003624 | 0·3 |
| 0·4 | ·023418 | ·019061 | ·015372 | ·012283 | ·009722 | ·007623 | ·005921 | ·004555 | ·003471 | 0·4 |
| 0·5 | ·021985 | ·017929 | ·014486 | ·011596 | ·009195 | ·007223 | ·005621 | ·004332 | ·003307 | 0·5 |
| 0·6 | ·020508 | ·016757 | ·013566 | ·010881 | ·008645 | ·006804 | ·005304 | ·004096 | ·003132 | 0·6 |
| 0·7 | ·019004 | ·015560 | ·012622 | ·010144 | ·008076 | ·006369 | ·004975 | ·003849 | ·002950 | 0·7 |
| 0·8 | ·017488 | ·014349 | ·011665 | ·009395 | ·007495 | ·005923 | ·004637 | ·003595 | ·002760 | 0·8 |
| 0·9 | ·015977 | ·013138 | ·010704 | ·008640 | ·006908 | ·005471 | ·004292 | ·003335 | ·002566 | 0·9 |
| 1·0 | ·014489 | ·011941 | ·009751 | ·007889 | ·006322 | ·005018 | ·003945 | ·003072 | ·002369 | 1·0 |
| 1·1 | ·013039 | ·010771 | ·008816 | ·007149 | ·005742 | ·004569 | ·003601 | ·002810 | ·002172 | 1·1 |
| 1·2 | ·011642 | ·009640 | ·007909 | ·006429 | ·005176 | ·004128 | ·003261 | ·002551 | ·001977 | 1·2 |
| 1·3 | ·010311 | ·008559 | ·007039 | ·005735 | ·004629 | ·003701 | ·002931 | ·002299 | ·001785 | 1·3 |
| 1·4 | ·009056 | ·007536 | ·006214 | ·005076 | ·004107 | ·003292 | ·002613 | ·002055 | ·001600 | 1·4 |
| 1·5 | ·007888 | ·006580 | ·005439 | ·004454 | ·003614 | ·002904 | ·002311 | ·001822 | ·001422 | 1·5 |
| 1·6 | ·006811 | ·005697 | ·004721 | ·003876 | ·003153 | ·002540 | ·002027 | ·001602 | ·001254 | 1·6 |
| 1·7 | ·005830 | ·004889 | ·004062 | ·003344 | ·002727 | ·002203 | ·001763 | ·001397 | ·001096 | 1·7 |
| 1·8 | ·004946 | ·004159 | ·003464 | ·002860 | ·002338 | ·001894 | ·001519 | ·001207 | ·000950 | 1·8 |
| 1·9 | ·004159 | ·003505 | ·002928 | ·002423 | ·001987 | ·001614 | ·001298 | ·001035 | ·000816 | 1·9 |
| 2·0 | ·003464 | ·002928 | ·002453 | ·002035 | ·001673 | ·001363 | ·001100 | ·000879 | ·000695 | 2·0 |
| 2·1 | ·002860 | ·002423 | ·002035 | ·001694 | ·001396 | ·001140 | ·000923 | ·000739 | ·000587 | 2·1 |
| 2·2 | ·002338 | ·001987 | ·001673 | ·001396 | ·001155 | ·000946 | ·000767 | ·000616 | ·000491 | 2·2 |
| 2·3 | ·001894 | ·001614 | ·001363 | ·001140 | ·000946 | ·000777 | ·000632 | ·000509 | ·000406 | 2·3 |
| 2·4 | ·001519 | ·001298 | ·001100 | ·000923 | ·000767 | ·000632 | ·000516 | ·000417 | ·000334 | 2·4 |
| 2·5 | ·001207 | ·001035 | ·000879 | ·000739 | ·000616 | ·000509 | ·000417 | ·000338 | ·000271 | 2·5 |
| 2·6 | ·000950 | ·000816 | ·000695 | ·000587 | ·000491 | ·000406 | ·000334 | ·000271 | ·000218 | 2·6 |

| $k$ | $r=\cdot40$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·315495 | ·295235 | ·274491 | ·253487 | ·232457 | ·211633 | ·191242 | ·171495 | ·152582 | 0·0 |
| 0·1 | ·295235 | ·276706 | ·257677 | ·238353 | ·218947 | ·199675 | ·180750 | ·162371 | ·144720 | 0·1 |
| 0·2 | ·274491 | ·257677 | ·240356 | ·222710 | ·204936 | ·187232 | ·169794 | ·152809 | ·136451 | 0·2 |
| 0·3 | ·253487 | ·238353 | ·222710 | ·206724 | ·190569 | ·174428 | ·158481 | ·142902 | ·127851 | 0·3 |
| 0·4 | ·232457 | ·218947 | ·204936 | ·190569 | ·176005 | ·161405 | ·146934 | ·132752 | ·119009 | 0·4 |
| 0·5 | ·211633 | ·199675 | ·187232 | ·174428 | ·161405 | ·148306 | ·135281 | ·122474 | ·110023 | 0·5 |
| 0·6 | ·191242 | ·180750 | ·169794 | ·158481 | ·146934 | ·135281 | ·123653 | ·112182 | ·100994 | 0·6 |
| 0·7 | ·171495 | ·162371 | ·152809 | ·142902 | ·132752 | ·122474 | ·112182 | ·101994 | ·092023 | 0·7 |
| 0·8 | ·152582 | ·144720 | ·136451 | ·127851 | ·119009 | ·110023 | ·100994 | ·092023 | ·083213 | 0·8 |
| 0·9 | ·134666 | ·127955 | ·120869 | ·113473 | ·105840 | ·098055 | ·090203 | ·082375 | ·074659 | 0·9 |
| 1·0 | ·117883 | ·112208 | ·106193 | ·099890 | ·093363 | ·086679 | ·079913 | ·073143 | ·066446 | 1·0 |
| 1·1 | ·102331 | ·097578 | ·092521 | ·087202 | ·081671 | ·075987 | ·070212 | ·064411 | ·058651 | 1·1 |
| 1·2 | ·088079 | ·084137 | ·079926 | ·075480 | ·070839 | ·066051 | ·061167 | ·056243 | ·051335 | 1·2 |
| 1·3 | ·075160 | ·071923 | ·068451 | ·064771 | ·060914 | ·056920 | ·052829 | ·048689 | ·044547 | 1·3 |
| 1·4 | ·063577 | ·060945 | ·058111 | ·055095 | ·051922 | ·048622 | ·045229 | ·041782 | ·038320 | 1·4 |
| 1·5 | ·053305 | ·051187 | ·048897 | ·046449 | ·043864 | ·041165 | ·038379 | ·035537 | ·032671 | 1·5 |
| 1·6 | ·044294 | ·042606 | ·040774 | ·038809 | ·036724 | ·034538 | ·032273 | ·029953 | ·027605 | 1·6 |
| 1·7 | ·036474 | ·035144 | ·033693 | ·032131 | ·030466 | ·028715 | ·026892 | ·025017 | ·023111 | 1·7 |
| 1·8 | ·029762 | ·028724 | ·027588 | ·026358 | ·025043 | ·023653 | ·022201 | ·020702 | ·019171 | 1·8 |
| 1·9 | ·024063 | ·023261 | ·022380 | ·021423 | ·020395 | ·019303 | ·018158 | ·016971 | ·015754 | 1·9 |
| 2·0 | ·019276 | ·018663 | ·017987 | ·017249 | ·016454 | ·015606 | ·014712 | ·013782 | ·012825 | 2·0 |
| 2·1 | ·015297 | ·014834 | ·014321 | ·013758 | ·013149 | ·012497 | ·011807 | ·011086 | ·010341 | 2·1 |
| 2·2 | ·012026 | ·011680 | ·011295 | ·010870 | ·010409 | ·009913 | ·009386 | ·008833 | ·008258 | 2·2 |
| 2·3 | ·009366 | ·009110 | ·008823 | ·008507 | ·008161 | ·007788 | ·007389 | ·006969 | ·006531 | 2·3 |
| 2·4 | ·007225 | ·007038 | ·006827 | ·006594 | ·006337 | ·006059 | ·005761 | ·005446 | ·005116 | 2·4 |
| 2·5 | ·005521 | ·005386 | ·005233 | ·005062 | ·004874 | ·004669 | ·004449 | ·004214 | ·003968 | 2·5 |
| 2·6 | ·004179 | ·004082 | ·003972 | ·003848 | ·003712 | ·003563 | ·003401 | ·003229 | ·003047 | 2·6 |

| $k$ | $r=\cdot40$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·134666 | ·117883 | ·102331 | ·088079 | ·075160 | ·063577 | ·053305 | ·044294 | ·036474 | 0·0 |
| 0·1 | ·127955 | ·112208 | ·097578 | ·084137 | ·071923 | ·060945 | ·051187 | ·042606 | ·035144 | 0·1 |
| 0·2 | ·120869 | ·106193 | ·092521 | ·079926 | ·068451 | ·058111 | ·048897 | ·040774 | ·033693 | 0·2 |
| 0·3 | ·113473 | ·099890 | ·087202 | ·075480 | ·064771 | ·055095 | ·046449 | ·038809 | ·032131 | 0·3 |
| 0·4 | ·105840 | ·093363 | ·081671 | ·070839 | ·060914 | ·051922 | ·043864 | ·036724 | ·030466 | 0·4 |
| 0·5 | ·098055 | ·086679 | ·075987 | ·066051 | ·056920 | ·048622 | ·041165 | ·034538 | ·028715 | 0·5 |
| 0·6 | ·090203 | ·079913 | ·070212 | ·061167 | ·052829 | ·045229 | ·038379 | ·032273 | ·026892 | 0·6 |
| 0·7 | ·082375 | ·073143 | ·064411 | ·056243 | ·048689 | ·041782 | ·035537 | ·029953 | ·025017 | 0·7 |
| 0·8 | ·074659 | ·066446 | ·058651 | ·051335 | ·044547 | ·038320 | ·032671 | ·027605 | ·023111 | 0·8 |
| 0·9 | ·067140 | ·059896 | ·052997 | ·046500 | ·040451 | ·034882 | ·029814 | ·025254 | ·021196 | 0·9 |
| 1·0 | ·059896 | ·053563 | ·047510 | ·041789 | ·036445 | ·031508 | ·026999 | ·022929 | ·019294 | 1·0 |
| 1·1 | ·052997 | ·047510 | ·042246 | ·037254 | ·032573 | ·028234 | ·024258 | ·020655 | ·017427 | 1·1 |
| 1·2 | ·046500 | ·041789 | ·037254 | ·032937 | ·028874 | ·025094 | ·021618 | ·018457 | ·015615 | 1·2 |
| 1·3 | ·040451 | ·036445 | ·032573 | ·028874 | ·025380 | ·022117 | ·019105 | ·016356 | ·013876 | 1·3 |
| 1·4 | ·034882 | ·031508 | ·028234 | ·025094 | ·022117 | ·019326 | ·016741 | ·014373 | ·012228 | 1·4 |
| 1·5 | ·029814 | ·026999 | ·024258 | ·021618 | ·019105 | ·016741 | ·014542 | ·012520 | ·010683 | 1·5 |
| 1·6 | ·025254 | ·022929 | ·020655 | ·018457 | ·016356 | ·014373 | ·012520 | ·010811 | ·009251 | 1·6 |
| 1·7 | ·021196 | ·019294 | ·017427 | ·015615 | ·013876 | ·012228 | ·010683 | ·009251 | ·007940 | 1·7 |
| 1·8 | ·017627 | ·016086 | ·014568 | ·013089 | ·011664 | ·010308 | ·009032 | ·007844 | ·006752 | 1·8 |
| 1·9 | ·014521 | ·013287 | ·012065 | ·010870 | ·009714 | ·008609 | ·007565 | ·006590 | ·005690 | 1·9 |
| 2·0 | ·011851 | ·010871 | ·009898 | ·008942 | ·008013 | ·007122 | ·006277 | ·005484 | ·004750 | 2·0 |
| 2·1 | ·009579 | ·008810 | ·008043 | ·007286 | ·006548 | ·005836 | ·005159 | ·004521 | ·003927 | 2·1 |
| 2·2 | ·007669 | ·007071 | ·006473 | ·005879 | ·005299 | ·004737 | ·004199 | ·003691 | ·003216 | 2·2 |
| 2·3 | ·006080 | ·005621 | ·005158 | ·004699 | ·004246 | ·003807 | ·003385 | ·002984 | ·002608 | 2·3 |
| 2·4 | ·004774 | ·004424 | ·004071 | ·003718 | ·003370 | ·003030 | ·002702 | ·002389 | ·002095 | 2·4 |
| 2·5 | ·003711 | ·003448 | ·003181 | ·002914 | ·002648 | ·002388 | ·002136 | ·001894 | ·001666 | 2·5 |
| 2·6 | ·002857 | ·002661 | ·002462 | ·002260 | ·002060 | ·001863 | ·001671 | ·001487 | ·001311 | 2·6 |

| $k$ | $r = \cdot 40$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 1 \cdot 8$ | $h = 1 \cdot 9$ | $h = 2 \cdot 0$ | $h = 2 \cdot 1$ | $h = 2 \cdot 2$ | $h = 2 \cdot 3$ | $h = 2 \cdot 4$ | $h = 2 \cdot 5$ | $h = 2 \cdot 6$ | |
| 0·0 | ·029762 | ·024063 | ·019276 | ·015297 | ·012026 | ·009366 | ·007225 | ·005521 | ·004179 | 0·0 |
| 0·1 | ·028724 | ·023261 | ·018663 | ·014834 | ·011680 | ·009110 | ·007038 | ·005386 | ·004082 | 0·1 |
| 0·2 | ·027588 | ·022380 | ·017987 | ·014321 | ·011295 | ·008823 | ·006827 | ·005233 | ·003972 | 0·2 |
| 0·3 | ·026358 | ·021423 | ·017249 | ·013758 | ·010870 | ·008507 | ·006594 | ·005062 | ·003848 | 0·3 |
| 0·4 | ·025043 | ·020395 | ·016454 | ·013149 | ·010409 | ·008161 | ·006337 | ·004874 | ·003712 | 0·4 |
| 0·5 | ·023653 | ·019303 | ·015606 | ·012497 | ·009913 | ·007788 | ·006059 | ·004669 | ·003563 | 0·5 |
| 0·6 | ·022201 | ·018158 | ·014712 | ·011807 | ·009386 | ·007389 | ·005761 | ·004449 | ·003401 | 0·6 |
| 0·7 | ·020702 | ·016971 | ·013782 | ·011086 | ·008833 | ·006969 | ·005446 | ·004214 | ·003229 | 0·7 |
| 0·8 | ·019171 | ·015754 | ·012825 | ·010341 | ·008258 | ·006531 | ·005116 | ·003968 | ·003047 | 0·8 |
| 0·9 | ·017627 | ·014521 | ·011851 | ·009579 | ·007669 | ·006080 | ·004774 | ·003711 | ·002857 | 0·9 |
| 1·0 | ·016086 | ·013287 | ·010871 | ·008810 | ·007071 | ·005621 | ·004424 | ·003448 | ·002661 | 1·0 |
| 1·1 | ·014568 | ·012065 | ·009898 | ·008043 | ·006473 | ·005158 | ·004071 | ·003181 | ·002462 | 1·1 |
| 1·2 | ·013089 | ·010870 | ·008942 | ·007286 | ·005879 | ·004699 | ·003718 | ·002914 | ·002260 | 1·2 |
| 1·3 | ·011664 | ·009714 | ·008013 | ·006548 | ·005299 | ·004246 | ·003370 | ·002648 | ·002060 | 1·3 |
| 1·4 | ·010308 | ·008609 | ·007122 | ·005836 | ·004737 | ·003807 | ·003030 | ·002388 | ·001863 | 1·4 |
| 1·5 | ·009032 | ·007565 | ·006277 | ·005159 | ·004199 | ·003385 | ·002702 | ·002136 | ·001671 | 1·5 |
| 1·6 | ·007844 | ·006590 | ·005484 | ·004521 | ·003691 | ·002984 | ·002389 | ·001894 | ·001487 | 1·6 |
| 1·7 | ·006752 | ·005690 | ·004750 | ·003927 | ·003216 | ·002608 | ·002095 | ·001666 | ·001311 | 1·7 |
| 1·8 | ·005760 | ·004868 | ·004076 | ·003381 | ·002777 | ·002260 | ·001820 | ·001452 | ·001147 | 1·8 |
| 1·9 | ·004868 | ·004127 | ·003467 | ·002884 | ·002377 | ·001940 | ·001568 | ·001255 | ·000994 | 1·9 |
| 2·0 | ·004076 | ·003467 | ·002921 | ·002438 | ·002015 | ·001650 | ·001338 | ·001074 | ·000854 | 2·0 |
| 2·1 | ·003381 | ·002884 | ·002438 | ·002041 | ·001693 | ·001391 | ·001131 | ·000911 | ·000727 | 2·1 |
| 2·2 | ·002777 | ·002377 | ·002015 | ·001693 | ·001409 | ·001161 | ·000948 | ·000766 | ·000613 | 2·2 |
| 2·3 | ·002260 | ·001940 | ·001650 | ·001391 | ·001161 | ·000960 | ·000786 | ·000637 | ·000512 | 2·3 |
| 2·4 | ·001820 | ·001568 | ·001338 | ·001131 | ·000948 | ·000786 | ·000646 | ·000525 | ·000423 | 2·4 |
| 2·5 | ·001452 | ·001255 | ·001074 | ·000911 | ·000766 | ·000637 | ·000525 | ·000429 | ·000347 | 2·5 |
| 2·6 | ·001147 | ·000994 | ·000854 | ·000727 | ·000613 | ·000512 | ·000423 | ·000347 | ·000281 | 2·6 |

| $k$ | $r = \cdot 45$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 0 \cdot 0$ | $h = 0 \cdot 1$ | $h = 0 \cdot 2$ | $h = 0 \cdot 3$ | $h = 0 \cdot 4$ | $h = 0 \cdot 5$ | $h = 0 \cdot 6$ | $h = 0 \cdot 7$ | $h = 0 \cdot 8$ | |
| 0·0 | ·324288 | ·303974 | ·283071 | ·261810 | ·240432 | ·219182 | ·198300 | ·178014 | ·158530 | 0·0 |
| 0·1 | ·303974 | ·285438 | ·266295 | ·246755 | ·227039 | ·207375 | ·187987 | ·169091 | ·150884 | 0·1 |
| 0·2 | ·283071 | ·266295 | ·248905 | ·231089 | ·213048 | ·194990 | ·177124 | ·159651 | ·142759 | 0·2 |
| 0·3 | ·261810 | ·246755 | ·231089 | ·214979 | ·198603 | ·182152 | ·165817 | ·149783 | ·134229 | 0·3 |
| 0·4 | ·240432 | ·227039 | ·213048 | ·198603 | ·183864 | ·169000 | ·154185 | ·139590 | ·125380 | 0·4 |
| 0·5 | ·219182 | ·207375 | ·194990 | ·182152 | ·169000 | ·155684 | ·142361 | ·129186 | ·116309 | 0·5 |
| 0·6 | ·198300 | ·187987 | ·177124 | ·165817 | ·154185 | ·142361 | ·130483 | ·118691 | ·107120 | 0·6 |
| 0·7 | ·178014 | ·169091 | ·159651 | ·149783 | ·139590 | ·129186 | ·118691 | ·108229 | ·097923 | 0·7 |
| 0·8 | ·158530 | ·150884 | ·142759 | ·134229 | ·125380 | ·116309 | ·107120 | ·097923 | ·088825 | 0·8 |
| 0·9 | ·140029 | ·133540 | ·126615 | ·119312 | ·111704 | ·103870 | ·095901 | ·087889 | ·079932 | 0·9 |
| 1·0 | ·122658 | ·117207 | ·111362 | ·105172 | ·098693 | ·091993 | ·085148 | ·078236 | ·071341 | 1·0 |
| 1·1 | ·106532 | ·101999 | ·097116 | ·091921 | ·086459 | ·080785 | ·074962 | ·069056 | ·063139 | 1·1 |
| 1·2 | ·091729 | ·087998 | ·083961 | ·079645 | ·075087 | ·070330 | ·065426 | ·060429 | ·055401 | 1·2 |
| 1·3 | ·078294 | ·075255 | ·071951 | ·068403 | ·064637 | ·060690 | ·056601 | ·052416 | ·048186 | 1·3 |
| 1·4 | ·066235 | ·063786 | ·061110 | ·058223 | ·055145 | ·051903 | ·048529 | ·045060 | ·041536 | 1·4 |
| 1·5 | ·055531 | ·053579 | ·051435 | ·049111 | ·046621 | ·043986 | ·041230 | ·038384 | ·035480 | 1·5 |
| 1·6 | ·046136 | ·044596 | ·042897 | ·041046 | ·039053 | ·036934 | ·034708 | ·032397 | ·030027 | 1·6 |
| 1·7 | ·037981 | ·036779 | ·035447 | ·033989 | ·032411 | ·030725 | ·028945 | ·027088 | ·025176 | 1·7 |
| 1·8 | ·030979 | ·030052 | ·029019 | ·027882 | ·026646 | ·025320 | ·023912 | ·022436 | ·020909 | 1·8 |
| 1·9 | ·025034 | ·024326 | ·023534 | ·022658 | ·021701 | ·020667 | ·019566 | ·018406 | ·017200 | 1·9 |
| 2·0 | ·020041 | ·019507 | ·018906 | ·018238 | ·017505 | ·016709 | ·015857 | ·014955 | ·014012 | 2·0 |
| 2·1 | ·015893 | ·015494 | ·015044 | ·014540 | ·013985 | ·013379 | ·012727 | ·012033 | ·011304 | 2·1 |
| 2·2 | ·012485 | ·012191 | ·011856 | ·011481 | ·011065 | ·010609 | ·010115 | ·009588 | ·009031 | 2·2 |
| 2·3 | ·009714 | ·009500 | ·009255 | ·008978 | ·008670 | ·008330 | ·007961 | ·007564 | ·007143 | 2·3 |
| 2·4 | ·007487 | ·007332 | ·007155 | ·006953 | ·006727 | ·006477 | ·006204 | ·005909 | ·005594 | 2·4 |
| 2·5 | ·005715 | ·005605 | ·005478 | ·005332 | ·005169 | ·004987 | ·004787 | ·004570 | ·004338 | 2·5 |
| 2·6 | ·004321 | ·004243 | ·004153 | ·004050 | ·003933 | ·003802 | ·003657 | ·003500 | ·003330 | 2·6 |

| $k$ | $r=\cdot45$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·140029 | ·122658 | ·106532 | ·091729 | ·078294 | ·066235 | ·055531 | ·046136 | ·037981 | $0\cdot0$ |
| $0\cdot1$ | ·133540 | ·117207 | ·101999 | ·087998 | ·075255 | ·063786 | ·053579 | ·044596 | ·036779 | $0\cdot1$ |
| $0\cdot2$ | ·126615 | ·111362 | ·097116 | ·083961 | ·071951 | ·061110 | ·051435 | ·042897 | ·035447 | $0\cdot2$ |
| $0\cdot3$ | ·119312 | ·105172 | ·091921 | ·079645 | ·068403 | ·058223 | ·049111 | ·041046 | ·033989 | $0\cdot3$ |
| $0\cdot4$ | ·111704 | ·098693 | ·086459 | ·075087 | ·064637 | ·055145 | ·046621 | ·039053 | ·032411 | $0\cdot4$ |
| $0\cdot5$ | ·103870 | ·091993 | ·080785 | ·070330 | ·060690 | ·051903 | ·043986 | ·036934 | ·030725 | $0\cdot5$ |
| $0\cdot6$ | ·095901 | ·085148 | ·074962 | ·065426 | ·056601 | ·048529 | ·041230 | ·034708 | ·028945 | $0\cdot6$ |
| $0\cdot7$ | ·087889 | ·078236 | ·069056 | ·060429 | ·052416 | ·045060 | ·038384 | ·032397 | ·027088 | $0\cdot7$ |
| $0\cdot8$ | ·079932 | ·071341 | ·063139 | ·055401 | ·048186 | ·041536 | ·035480 | ·030027 | ·025176 | $0\cdot8$ |
| $0\cdot9$ | ·072120 | ·064543 | ·057280 | ·050400 | ·043959 | ·038000 | ·032551 | ·027627 | ·023229 | $0\cdot9$ |
| $1\cdot0$ | ·064543 | ·057922 | ·051549 | ·045486 | ·039787 | ·034494 | ·029634 | ·025225 | ·021272 | $1\cdot0$ |
| $1\cdot1$ | ·057280 | ·051549 | ·046008 | ·040715 | ·035719 | ·031059 | ·026763 | ·022850 | ·019327 | $1\cdot1$ |
| $1\cdot2$ | ·050400 | ·045486 | ·040715 | ·036138 | ·031798 | ·027734 | ·023971 | ·020529 | ·017418 | $1\cdot2$ |
| $1\cdot3$ | ·043959 | ·039787 | ·035719 | ·031798 | ·028066 | ·024554 | ·021289 | ·018290 | ·015568 | $1\cdot3$ |
| $1\cdot4$ | ·038000 | ·034494 | ·031059 | ·027734 | ·024554 | ·021549 | ·018743 | ·016154 | ·013795 | $1\cdot4$ |
| $1\cdot5$ | ·032551 | ·029634 | ·026763 | ·023971 | ·021289 | ·018743 | ·016355 | ·014142 | ·012116 | $1\cdot5$ |
| $1\cdot6$ | ·027627 | ·025225 | ·022850 | ·020529 | ·018290 | ·016154 | ·014142 | ·012269 | ·010547 | $1\cdot6$ |
| $1\cdot7$ | ·023229 | ·021272 | ·019327 | ·017418 | ·015568 | ·013795 | ·012116 | ·010547 | ·009097 | $1\cdot7$ |
| $1\cdot8$ | ·019347 | ·017769 | ·016193 | ·014639 | ·013126 | ·011669 | ·010283 | ·008981 | ·007773 | $1\cdot8$ |
| $1\cdot9$ | ·015960 | ·014700 | ·013438 | ·012186 | ·010961 | ·009776 | ·008644 | ·007576 | ·006579 | $1\cdot9$ |
| $2\cdot0$ | ·013038 | ·012045 | ·011043 | ·010046 | ·009065 | ·008112 | ·007197 | ·006329 | ·005515 | $2\cdot0$ |
| $2\cdot1$ | ·010548 | ·009772 | ·008987 | ·008201 | ·007424 | ·006665 | ·005933 | ·005235 | ·004578 | $2\cdot1$ |
| $2\cdot2$ | ·008450 | ·007851 | ·007241 | ·006628 | ·006019 | ·005422 | ·004843 | ·004288 | ·003763 | $2\cdot2$ |
| $2\cdot3$ | ·006702 | ·006245 | ·005777 | ·005304 | ·004832 | ·004367 | ·003913 | ·003477 | ·003062 | $2\cdot3$ |
| $2\cdot4$ | ·005263 | ·004917 | ·004562 | ·004202 | ·003840 | ·003482 | ·003131 | ·002791 | ·002467 | $2\cdot4$ |
| $2\cdot5$ | ·004091 | ·003833 | ·003567 | ·003295 | ·003021 | ·002748 | ·002479 | ·002218 | ·001967 | $2\cdot5$ |
| $2\cdot6$ | ·003149 | ·002958 | ·002761 | ·002558 | ·002352 | ·002147 | ·001943 | ·001744 | ·001553 | $2\cdot6$ |

| $k$ | $r=\cdot45$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·030979 | ·025034 | ·020041 | ·015893 | ·012485 | ·009714 | ·007487 | ·005715 | ·004321 | $0\cdot0$ |
| $0\cdot1$ | ·030052 | ·024326 | ·019507 | ·015494 | ·012191 | ·009500 | ·007332 | ·005605 | ·004243 | $0\cdot1$ |
| $0\cdot2$ | ·029019 | ·023534 | ·018906 | ·015044 | ·011856 | ·009255 | ·007155 | ·005478 | ·004153 | $0\cdot2$ |
| $0\cdot3$ | ·027882 | ·022658 | ·018238 | ·014540 | ·011481 | ·008978 | ·006953 | ·005332 | ·004050 | $0\cdot3$ |
| $0\cdot4$ | ·026646 | ·021701 | ·017505 | ·013985 | ·011065 | ·008670 | ·006727 | ·005169 | ·003933 | $0\cdot4$ |
| $0\cdot5$ | ·025320 | ·020667 | ·016709 | ·013379 | ·010609 | ·008330 | ·006477 | ·004987 | ·003802 | $0\cdot5$ |
| $0\cdot6$ | ·023912 | ·019566 | ·015857 | ·012727 | ·010115 | ·007961 | ·006204 | ·004787 | ·003657 | $0\cdot6$ |
| $0\cdot7$ | ·022436 | ·018406 | ·014955 | ·012033 | ·009588 | ·007564 | ·005909 | ·004570 | ·003500 | $0\cdot7$ |
| $0\cdot8$ | ·020909 | ·017200 | ·014012 | ·011304 | ·009031 | ·007143 | ·005594 | ·004338 | ·003330 | $0\cdot8$ |
| $0\cdot9$ | ·019347 | ·015960 | ·013038 | ·010548 | ·008450 | ·006702 | ·005263 | ·004091 | ·003149 | $0\cdot9$ |
| $1\cdot0$ | ·017769 | ·014700 | ·012045 | ·009772 | ·007851 | ·006245 | ·004917 | ·003833 | ·002958 | $1\cdot0$ |
| $1\cdot1$ | ·016193 | ·013438 | ·011043 | ·008987 | ·007241 | ·005777 | ·004562 | ·003567 | ·002761 | $1\cdot1$ |
| $1\cdot2$ | ·014639 | ·012186 | ·010046 | ·008201 | ·006628 | ·005304 | ·004202 | ·003295 | ·002558 | $1\cdot2$ |
| $1\cdot3$ | ·013126 | ·010961 | ·009065 | ·007424 | ·006019 | ·004832 | ·003840 | ·003021 | ·002352 | $1\cdot3$ |
| $1\cdot4$ | ·011669 | ·009776 | ·008112 | ·006665 | ·005422 | ·004367 | ·003482 | ·002748 | ·002147 | $1\cdot4$ |
| $1\cdot5$ | ·010283 | ·008644 | ·007197 | ·005933 | ·004843 | ·003913 | ·003131 | ·002479 | ·001943 | $1\cdot5$ |
| $1\cdot6$ | ·008981 | ·007576 | ·006329 | ·005235 | ·004288 | ·003477 | ·002791 | ·002218 | ·001744 | $1\cdot6$ |
| $1\cdot7$ | ·007773 | ·006579 | ·005515 | ·004578 | ·003763 | ·003062 | ·002467 | ·001967 | ·001553 | $1\cdot7$ |
| $1\cdot8$ | ·006665 | ·005661 | ·004763 | ·003968 | ·003273 | ·002673 | ·002161 | ·001729 | ·001370 | $1\cdot8$ |
| $1\cdot9$ | ·005661 | ·004826 | ·004074 | ·003406 | ·002820 | ·002312 | ·001876 | ·001507 | ·001198 | $1\cdot9$ |
| $2\cdot0$ | ·004763 | ·004074 | ·003452 | ·002897 | ·002407 | ·001980 | ·001613 | ·001300 | ·001038 | $2\cdot0$ |
| $2\cdot1$ | ·003968 | ·003406 | ·002897 | ·002440 | ·002035 | ·001680 | ·001374 | ·001112 | ·000890 | $2\cdot1$ |
| $2\cdot2$ | ·003273 | ·002820 | ·002407 | ·002035 | ·001703 | ·001412 | ·001159 | ·000941 | ·000757 | $2\cdot2$ |
| $2\cdot3$ | ·002673 | ·002312 | ·001980 | ·001680 | ·001412 | ·001175 | ·000968 | ·000789 | ·000637 | $2\cdot3$ |
| $2\cdot4$ | ·002161 | ·001876 | ·001613 | ·001374 | ·001159 | ·000968 | ·000800 | ·000655 | ·000531 | $2\cdot4$ |
| $2\cdot5$ | ·001729 | ·001507 | ·001300 | ·001112 | ·000941 | ·000789 | ·000655 | ·000539 | ·000438 | $2\cdot5$ |
| $2\cdot6$ | ·001370 | ·001198 | ·001038 | ·000890 | ·000757 | ·000637 | ·000531 | ·000438 | ·000358 | $2\cdot6$ |

| $k$ | $r=\cdot50$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·333333 | ·312961 | ·291886 | ·270344 | ·248589 | ·226878 | ·205468 | ·184605 | ·164512 | 0·0 |
| 0·1 | ·312961 | ·294422 | ·275161 | ·255392 | ·235345 | ·215260 | ·195377 | ·175927 | ·157126 | 0·1 |
| 0·2 | ·291886 | ·275161 | ·257709 | ·239718 | ·221397 | ·202965 | ·184644 | ·166650 | ·149190 | 0·2 |
| 0·3 | ·270344 | ·255392 | ·239718 | ·223488 | ·206888 | ·190114 | ·173370 | ·156858 | ·140769 | 0·3 |
| 0·4 | ·248589 | ·235345 | ·221397 | ·206888 | ·191979 | ·176847 | ·161676 | ·146649 | ·131946 | 0·4 |
| 0·5 | ·226878 | ·215260 | ·202965 | ·190114 | ·176847 | ·163320 | ·149694 | ·136139 | ·122816 | 0·5 |
| 0·6 | ·205468 | ·195377 | ·184644 | ·173370 | ·161676 | ·149694 | ·137570 | ·125451 | ·113486 | 0·6 |
| 0·7 | ·184605 | ·175927 | ·166650 | ·156858 | ·146649 | ·136139 | ·125451 | ·114718 | ·104071 | 0·7 |
| 0·8 | ·164512 | ·157126 | ·149190 | ·140769 | ·131946 | ·122816 | ·113486 | ·104071 | ·094686 | 0·8 |
| 0·9 | ·145388 | ·139168 | ·132448 | ·125281 | ·117733 | ·109882 | ·101819 | ·093640 | ·085448 | 0·9 |
| 1·0 | ·127398 | ·122215 | ·116586 | ·110550 | ·104159 | ·097477 | ·090578 | ·083546 | ·076465 | 1·0 |
| 1·1 | ·110671 | ·106398 | ·101733 | ·096704 | ·091350 | ·085722 | ·079882 | ·073896 | ·067839 | 1·1 |
| 1·2 | ·095297 | ·091814 | ·087990 | ·083844 | ·079407 | ·074718 | ·069825 | ·064784 | ·059656 | 1·2 |
| 1·3 | ·081329 | ·078522 | ·075421 | ·072042 | ·068404 | ·064539 | ·060485 | ·056284 | ·051988 | 1·3 |
| 1·4 | ·068785 | ·066546 | ·064061 | ·061337 | ·058388 | ·055237 | ·051913 | ·048451 | ·044891 | 1·4 |
| 1·5 | ·057646 | ·055882 | ·053912 | ·051740 | ·049377 | ·046836 | ·044142 | ·041320 | ·038401 | 1·5 |
| 1·6 | ·047867 | ·046493 | ·044950 | ·043238 | ·041365 | ·039340 | ·037180 | ·034905 | ·032539 | 1·6 |
| 1·7 | ·039379 | ·038321 | ·037126 | ·035793 | ·034325 | ·032729 | ·031018 | ·029204 | ·027307 | 1·7 |
| 1·8 | ·032095 | ·031290 | ·030375 | ·029348 | ·028211 | ·026968 | ·025627 | ·024198 | ·022695 | 1·8 |
| 1·9 | ·025912 | ·025307 | ·024615 | ·023834 | ·022963 | ·022006 | ·020968 | ·019854 | ·018677 | 1·9 |
| 2·0 | ·020724 | ·020274 | ·019756 | ·019169 | ·018510 | ·017782 | ·016987 | ·016130 | ·015218 | 2·0 |
| 2·1 | ·016417 | ·016087 | ·015704 | ·015268 | ·014776 | ·014228 | ·013627 | ·012974 | ·012276 | 2·1 |
| 2·2 | ·012882 | ·012642 | ·012363 | ·012043 | ·011679 | ·011272 | ·010823 | ·010332 | ·009804 | 2·2 |
| 2·3 | ·010011 | ·009840 | ·009638 | ·009406 | ·009141 | ·008842 | ·008510 | ·008145 | ·007750 | 2·3 |
| 2·4 | ·007706 | ·007585 | ·007441 | ·007275 | ·007083 | ·006867 | ·006624 | ·006357 | ·006065 | 2·4 |
| 2·5 | ·005875 | ·005790 | ·005689 | ·005571 | ·005435 | ·005280 | ·005105 | ·004911 | ·004698 | 2·5 |
| 2·6 | ·004436 | ·004377 | ·004307 | ·004225 | ·004129 | ·004019 | ·003894 | ·003755 | ·003602 | 2·6 |

| $k$ | $r=\cdot50$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·145388 | ·127398 | ·110671 | ·095297 | ·081329 | ·068785 | ·057646 | ·047867 | ·039379 | 0·0 |
| 0·1 | ·139168 | ·122215 | ·106398 | ·091814 | ·078522 | ·066546 | ·055882 | ·046493 | ·038321 | 0·1 |
| 0·2 | ·132448 | ·116586 | ·101733 | ·087990 | ·075421 | ·064061 | ·053912 | ·044950 | ·037126 | 0·2 |
| 0·3 | ·125281 | ·110550 | ·096704 | ·083844 | ·072042 | ·061337 | ·051740 | ·043238 | ·035793 | 0·3 |
| 0·4 | ·117733 | ·104159 | ·091350 | ·079407 | ·068404 | ·058388 | ·049377 | ·041365 | ·034325 | 0·4 |
| 0·5 | ·109882 | ·097477 | ·085722 | ·074718 | ·064539 | ·055237 | ·046836 | ·039340 | ·032729 | 0·5 |
| 0·6 | ·101819 | ·090578 | ·079882 | ·069825 | ·060485 | ·051913 | ·044142 | ·037180 | ·031018 | 0·6 |
| 0·7 | ·093640 | ·083546 | ·073896 | ·064784 | ·056284 | ·048451 | ·041320 | ·034905 | ·029204 | 0·7 |
| 0·8 | ·085448 | ·076465 | ·067839 | ·059656 | ·051988 | ·044891 | ·038401 | ·032539 | ·027307 | 0·8 |
| 0·9 | ·077344 | ·069426 | ·061786 | ·054504 | ·047649 | ·041275 | ·035421 | ·030110 | ·025349 | 0·9 |
| 1·0 | ·069426 | ·062514 | ·055812 | ·049394 | ·043323 | ·037651 | ·032418 | ·027648 | ·023353 | 1·0 |
| 1·1 | ·061786 | ·055812 | ·049991 | ·044388 | ·039062 | ·034063 | ·029428 | ·025184 | ·021345 | 1·1 |
| 1·2 | ·054504 | ·049394 | ·044388 | ·039545 | ·034920 | ·030556 | ·026490 | ·022749 | ·019349 | 1·2 |
| 1·3 | ·047649 | ·043323 | ·039062 | ·034920 | ·030942 | ·027170 | ·023639 | ·020373 | ·017391 | 1·3 |
| 1·4 | ·041275 | ·037651 | ·034063 | ·030556 | ·027170 | ·023944 | ·020907 | ·018085 | ·015494 | 1·4 |
| 1·5 | ·035421 | ·032418 | ·029428 | ·026490 | ·023639 | ·020907 | ·018323 | ·015909 | ·013681 | 1·5 |
| 1·6 | ·030110 | ·027648 | ·025184 | ·022749 | ·020373 | ·018085 | ·015909 | ·013864 | ·011969 | 1·6 |
| 1·7 | ·025349 | ·023353 | ·021345 | ·019349 | ·017391 | ·015494 | ·013681 | ·011969 | ·010372 | 1·7 |
| 1·8 | ·021134 | ·019534 | ·017915 | ·016297 | ·014701 | ·013146 | ·011652 | ·010233 | ·008902 | 1·8 |
| 1·9 | ·017447 | ·016178 | ·014888 | ·013591 | ·012304 | ·011044 | ·009826 | ·008663 | ·007566 | 1·9 |
| 2·0 | ·014260 | ·013266 | ·012249 | ·011221 | ·010196 | ·009186 | ·008204 | ·007261 | ·006367 | 2·0 |
| 2·1 | ·011539 | ·010769 | ·009977 | ·009171 | ·008363 | ·007563 | ·006780 | ·006024 | ·005304 | 2·1 |
| 2·2 | ·009242 | ·008654 | ·008043 | ·007420 | ·006790 | ·006163 | ·005546 | ·004947 | ·004373 | 2·2 |
| 2·3 | ·007328 | ·006883 | ·006419 | ·005941 | ·005457 | ·004971 | ·004490 | ·004021 | ·003568 | 2·3 |
| 2·4 | ·005751 | ·005418 | ·005069 | ·004708 | ·004339 | ·003968 | ·003598 | ·003234 | ·002882 | 2·4 |
| 2·5 | ·004468 | ·004222 | ·003962 | ·003692 | ·003415 | ·003134 | ·002852 | ·002574 | ·002303 | 2·5 |
| 2·6 | ·003435 | ·003255 | ·003065 | ·002865 | ·002659 | ·002449 | ·002237 | ·002027 | ·001820 | 2·6 |

| $k$ | $r=\cdot50$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·032095 | ·025912 | ·020724 | ·016417 | ·012882 | ·010011 | ·007706 | ·005875 | ·004436 | $0\cdot0$ |
| $0\cdot1$ | ·031290 | ·025307 | ·020274 | ·016087 | ·012642 | ·009840 | ·007585 | ·005790 | ·004377 | $0\cdot1$ |
| $0\cdot2$ | ·030375 | ·024615 | ·019756 | ·015704 | ·012363 | ·009638 | ·007441 | ·005689 | ·004307 | $0\cdot2$ |
| $0\cdot3$ | ·029348 | ·023834 | ·019169 | ·015268 | ·012043 | ·009406 | ·007275 | ·005571 | ·004225 | $0\cdot3$ |
| $0\cdot4$ | ·028211 | ·022963 | ·018510 | ·014776 | ·011679 | ·009141 | ·007083 | ·005435 | ·004129 | $0\cdot4$ |
| $0\cdot5$ | ·026968 | ·022006 | ·017782 | ·014228 | ·011272 | ·008842 | ·006867 | ·005280 | ·004019 | $0\cdot5$ |
| $0\cdot6$ | ·025627 | ·020968 | ·016987 | ·013627 | ·010823 | ·008510 | ·006624 | ·005105 | ·003894 | $0\cdot6$ |
| $0\cdot7$ | ·024198 | ·019854 | ·016130 | ·012974 | ·010332 | ·008145 | ·006357 | ·004911 | ·003755 | $0\cdot7$ |
| $0\cdot8$ | ·022695 | ·018677 | ·015218 | ·012276 | ·009804 | ·007750 | ·006065 | ·004698 | ·003602 | $0\cdot8$ |
| $0\cdot9$ | ·021134 | ·017447 | ·014260 | ·011539 | ·009242 | ·007328 | ·005751 | ·004468 | ·003435 | $0\cdot9$ |
| $1\cdot0$ | ·019534 | ·016178 | ·013266 | ·010769 | ·008654 | ·006883 | ·005418 | ·004222 | ·003255 | $1\cdot0$ |
| $1\cdot1$ | ·017915 | ·014888 | ·012249 | ·009977 | ·008043 | ·006419 | ·005069 | ·003962 | ·003065 | $1\cdot1$ |
| $1\cdot2$ | ·016297 | ·013591 | ·011221 | ·009171 | ·007420 | ·005941 | ·004708 | ·003692 | ·002865 | $1\cdot2$ |
| $1\cdot3$ | ·014701 | ·012304 | ·010196 | 008363 | ·006790 | ·005457 | ·004339 | ·003415 | ·002659 | $1\cdot3$ |
| $1\cdot4$ | ·013146 | ·011044 | ·009186 | ·007563 | ·006163 | ·004971 | ·003968 | ·003134 | ·002449 | $1\cdot4$ |
| $1\cdot5$ | ·011652 | ·009826 | ·008204 | ·006780 | ·005546 | ·004490 | ·003598 | ·002852 | ·002237 | $1\cdot5$ |
| $1\cdot6$ | ·010233 | ·008663 | ·007261 | ·006024 | ·004947 | ·004021 | ·003234 | ·002574 | ·002027 | $1\cdot6$ |
| $1\cdot7$ | ·008902 | ·007566 | ·006367 | ·005304 | ·004373 | ·003568 | ·002882 | ·002303 | ·001820 | $1\cdot7$ |
| $1\cdot8$ | ·007671 | ·006546 | ·005531 | ·004626 | ·003830 | ·003138 | ·002544 | ·002041 | ·001620 | $1\cdot8$ |
| $1\cdot9$ | ·006546 | ·005608 | ·004758 | ·003996 | ·003322 | ·002733 | ·002225 | ·001793 | ·001429 | $1\cdot9$ |
| $2\cdot0$ | ·005531 | ·004758 | ·004053 | ·003418 | ·002853 | ·002358 | ·001928 | ·001560 | ·001249 | $2\cdot0$ |
| $2\cdot1$ | ·004626 | ·003996 | ·003418 | ·002895 | ·002427 | ·002014 | ·001654 | ·001344 | ·001081 | $2\cdot1$ |
| $2\cdot2$ | ·003830 | ·003322 | ·002853 | ·002427 | ·002043 | ·001703 | ·001405 | ·001146 | ·000926 | $2\cdot2$ |
| $2\cdot3$ | ·003138 | ·002733 | ·002358 | ·002014 | ·001703 | ·001426 | ·001181 | ·000968 | ·000785 | $2\cdot3$ |
| $2\cdot4$ | ·002544 | ·002225 | ·001928 | ·001654 | ·001405 | ·001181 | ·000983 | ·000809 | ·000659 | $2\cdot4$ |
| $2\cdot5$ | ·002041 | ·001793 | ·001560 | ·001344 | ·001146 | ·000968 | ·000809 | ·000669 | ·000548 | $2\cdot5$ |
| $2\cdot6$ | ·001620 | ·001429 | ·001249 | ·001081 | ·000926 | ·000785 | ·000659 | ·000548 | ·000451 | $2\cdot6$ |

| $k$ | $r=\cdot55$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·342686 | ·322250 | ·300984 | ·279133 | ·256963 | ·234747 | ·212762 | ·191272 | ·170523 | $0\cdot0$ |
| $0\cdot1$ | ·322250 | ·303714 | ·284328 | ·264313 | ·243907 | ·223364 | ·202942 | ·182893 | ·163452 | $0\cdot1$ |
| $0\cdot2$ | ·300984 | ·284328 | ·266819 | ·248648 | ·230030 | ·211196 | ·192384 | ·173829 | ·155757 | $0\cdot2$ |
| $0\cdot3$ | ·279133 | ·264313 | ·248648 | ·232305 | ·215474 | ·198360 | ·181181 | ·164155 | ·147493 | $0\cdot3$ |
| $0\cdot4$ | ·256963 | ·243907 | ·230030 | ·215474 | ·200[illegible]91 | ·184994 | ·169449 | ·153964 | ·138735 | $0\cdot4$ |
| $0\cdot5$ | ·234747 | ·223364 | ·211196 | ·198360 | ·18[illegible]4 | ·171258 | ·157324 | ·143371 | ·129577 | $0\cdot5$ |
| $0\cdot6$ | ·212762 | ·202942 | ·192384 | ·181181 | ·169449 | ·157324 | ·144956 | ·132503 | ·120128 | $0\cdot6$ |
| $0\cdot7$ | ·191272 | ·182893 | ·173829 | ·164155 | ·153964 | ·143371 | ·132503 | ·121500 | ·110505 | $0\cdot7$ |
| $0\cdot8$ | ·170523 | ·163452 | ·155757 | ·147493 | ·138735 | ·129577 | ·120128 | ·110505 | ·100834 | $0\cdot8$ |
| $0\cdot9$ | ·150733 | ·144834 | ·138373 | ·131391 | ·123947 | ·116116 | ·107987 | ·099659 | ·091241 | $0\cdot9$ |
| $1\cdot0$ | ·132086 | ·127220 | ·121858 | ·116026 | ·109769 | ·103146 | ·096229 | ·089099 | ·081849 | $1\cdot0$ |
| $1\cdot1$ | ·114725 | ·110759 | ·106360 | ·101545 | ·096345 | ·090807 | ·084986 | ·078950 | ·072774 | $1\cdot1$ |
| $1\cdot2$ | ·098756 | ·095562 | ·091995 | ·088065 | ·083794 | ·079216 | ·074374 | ·069320 | ·064118 | $1\cdot2$ |
| $1\cdot3$ | ·084240 | ·081698 | ·078840 | ·075671 | ·072205 | ·068464 | ·064482 | ·060300 | ·055967 | $1\cdot3$ |
| $1\cdot4$ | ·071200 | ·069202 | ·066940 | ·064415 | ·061634 | ·058614 | ·055378 | ·051957 | ·048390 | $1\cdot4$ |
| $1\cdot5$ | ·059623 | ·058071 | ·056303 | ·054315 | ·052112 | ·049703 | ·047104 | ·044339 | ·041436 | $1\cdot5$ |
| $1\cdot6$ | ·049463 | ·048273 | ·046908 | ·045363 | ·043638 | ·041739 | ·039678 | ·037469 | ·035135 | $1\cdot6$ |
| $1\cdot7$ | ·040650 | ·039749 | ·038708 | ·037522 | ·036188 | ·034710 | ·033095 | ·031352 | ·029498 | $1\cdot7$ |
| $1\cdot8$ | ·033093 | ·032419 | ·031635 | ·030736 | ·029718 | ·028581 | ·027331 | ·025972 | ·024518 | $1\cdot8$ |
| $1\cdot9$ | ·026686 | ·026188 | ·025605 | ·024932 | ·024164 | ·023302 | ·022346 | ·021300 | ·020173 | $1\cdot9$ |
| $2\cdot0$ | ·021314 | ·020952 | ·020524 | ·020026 | ·019455 | ·018808 | ·018087 | ·017292 | ·016429 | $2\cdot0$ |
| $2\cdot1$ | ·016862 | ·016601 | ·016291 | ·015928 | ·015508 | ·015030 | ·014492 | ·013895 | ·013243 | $2\cdot1$ |
| $2\cdot2$ | ·013213 | ·013027 | ·012806 | ·012544 | ·012240 | ·011890 | ·011494 | ·011052 | ·010565 | $2\cdot2$ |
| $2\cdot3$ | ·010254 | ·010124 | ·009968 | ·009782 | ·009564 | ·009311 | ·009024 | ·008700 | ·008342 | $2\cdot3$ |
| $2\cdot4$ | ·007882 | ·007792 | ·007683 | ·007552 | ·007398 | ·007219 | ·007012 | ·006779 | ·006518 | $2\cdot4$ |
| $2\cdot5$ | ·006000 | ·005939 | ·005864 | ·005773 | ·005666 | ·005540 | ·005394 | ·005227 | ·005040 | $2\cdot5$ |
| $2\cdot6$ | ·004524 | ·004483 | ·004432 | ·004370 | ·004296 | ·004209 | ·004107 | ·003990 | ·003857 | $2\cdot6$ |

| $k$ | $r=\cdot55$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·150733 | ·132086 | ·114725 | ·098756 | ·084240 | ·071200 | ·059623 | ·049463 | ·040650 | 0·0 |
| 0·1 | ·144834 | ·127220 | ·110759 | ·095562 | ·081698 | ·069202 | ·058071 | ·048273 | ·039749 | 0·1 |
| 0·2 | ·138373 | ·121858 | ·106360 | ·091995 | ·078840 | ·066940 | ·056303 | ·046908 | ·038708 | 0·2 |
| 0·3 | ·131391 | ·116026 | ·101545 | ·088065 | ·075671 | ·064415 | ·054315 | ·045363 | ·037522 | 0·3 |
| 0·4 | ·123947 | ·109769 | ·096345 | ·083794 | ·072205 | ·061634 | ·052112 | ·043638 | ·036188 | 0·4 |
| 0·5 | ·116116 | ·103146 | ·090807 | ·079216 | ·068464 | ·058614 | ·049703 | ·041739 | ·034710 | 0·5 |
| 0·6 | ·107987 | ·096229 | ·084986 | ·074374 | ·064482 | ·055378 | ·047104 | ·039678 | ·033095 | 0·6 |
| 0·7 | ·099659 | ·089099 | ·078950 | ·069320 | ·060300 | ·051957 | ·044339 | ·037469 | ·031352 | 0·7 |
| 0·8 | ·091241 | ·081849 | ·072774 | ·064118 | ·055967 | ·048390 | ·041436 | ·035135 | ·029498 | 0·8 |
| 0·9 | ·082843 | ·074574 | ·066539 | ·058833 | ·051537 | ·044720 | ·038431 | ·032703 | ·027553 | 0·9 |
| 1·0 | ·074574 | ·067369 | ·060327 | ·053536 | ·047070 | ·040995 | ·035360 | ·030202 | ·025540 | 1·0 |
| 1·1 | ·066539 | ·060327 | ·054221 | ·048296 | ·042624 | ·037265 | ·032266 | ·027665 | ·023484 | 1·1 |
| 1·2 | ·058833 | ·053536 | ·048296 | ·043183 | ·038259 | ·033579 | ·029190 | ·025127 | ·021414 | 1·2 |
| 1·3 | ·051537 | ·047070 | ·042624 | ·038259 | ·034030 | ·029987 | ·026172 | ·022620 | ·019357 | 1·3 |
| 1·4 | ·044720 | ·040995 | ·037265 | ·033579 | ·029987 | ·026531 | ·023251 | ·020180 | ·017340 | 1·4 |
| 1·5 | ·038431 | ·035360 | ·032266 | ·029190 | ·026172 | ·023251 | ·020462 | ·017834 | ·015390 | 1·5 |
| 1·6 | ·032703 | ·030202 | ·027665 | ·025127 | ·022620 | ·020180 | ·017834 | ·015610 | ·013530 | 1·6 |
| 1·7 | ·027553 | ·025540 | ·023484 | ·021414 | ·019357 | ·017340 | ·015390 | ·013530 | ·011778 | 1·7 |
| 1·8 | ·022981 | ·021380 | ·019735 | ·018066 | ·016397 | ·014751 | ·013148 | ·011609 | ·010151 | 1·8 |
| 1·9 | ·018974 | ·017716 | ·016414 | ·015086 | ·013748 | ·012420 | ·011118 | ·009860 | ·008661 | 1·9 |
| 2·0 | ·015505 | ·014529 | ·013512 | ·012467 | ·011408 | ·010349 | ·009304 | ·008288 | ·007313 | 2·0 |
| 2·1 | ·012540 | ·011792 | ·011007 | ·010195 | ·009367 | ·008533 | ·007704 | ·006893 | ·006110 | 2·1 |
| 2·2 | ·010037 | ·009471 | ·008873 | ·008250 | ·007610 | ·006961 | ·006312 | ·005673 | ·005051 | 2·2 |
| 2·3 | ·007950 | ·007527 | ·007077 | ·006605 | ·006116 | ·005618 | ·005116 | ·004618 | ·004130 | 2·3 |
| 2·4 | ·006231 | ·005919 | ·005585 | ·005231 | ·004863 | ·004485 | ·004102 | ·003719 | ·003341 | 2·4 |
| 2·5 | ·004832 | ·004605 | ·004360 | ·004099 | ·003825 | ·003542 | ·003253 | ·002962 | ·002673 | 2·5 |
| 2·6 | ·003709 | ·003545 | ·003368 | ·003177 | ·002976 | ·002767 | ·002552 | ·002333 | ·002115 | 2·6 |

| $k$ | $r=\cdot55$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·033093 | ·026686 | ·021314 | ·016862 | ·013213 | ·010254 | ·007882 | ·006000 | ·004524 | 0·0 |
| 0·1 | ·032419 | ·026188 | ·020952 | ·016601 | ·013027 | ·010124 | ·007792 | ·005939 | ·004483 | 0·1 |
| 0·2 | ·031635 | ·025605 | ·020524 | ·016291 | ·012806 | ·009968 | ·007683 | ·005864 | ·004432 | 0·2 |
| 0·3 | ·030736 | ·024932 | ·020026 | ·015928 | ·012544 | ·009782 | ·007552 | ·005773 | ·004370 | 0·3 |
| 0·4 | ·029718 | ·024164 | ·019455 | ·015508 | ·012240 | ·009564 | ·007398 | ·005666 | ·004296 | 0·4 |
| 0·5 | ·028581 | ·023302 | ·018808 | ·015030 | ·011890 | ·009311 | ·007219 | ·005540 | ·004209 | 0·5 |
| 0·6 | ·027331 | ·022346 | ·018087 | ·014492 | ·011494 | ·009024 | ·007012 | ·005394 | ·004107 | 0·6 |
| 0·7 | ·025972 | ·021300 | ·017292 | ·013895 | ·011052 | ·008700 | ·006779 | ·005227 | ·003990 | 0·7 |
| 0·8 | ·024518 | ·020173 | ·016429 | ·013243 | ·010565 | ·008342 | ·006518 | ·005040 | ·003857 | 0·8 |
| 0·9 | ·022981 | ·018974 | ·015505 | ·012540 | ·010037 | ·007950 | ·006231 | ·004832 | ·003709 | 0·9 |
| 1·0 | ·021380 | ·017716 | ·014529 | ·011792 | ·009471 | ·007527 | ·005919 | ·004605 | ·003545 | 1·0 |
| 1·1 | ·019735 | ·016414 | ·013512 | ·011007 | ·008873 | ·007077 | ·005585 | ·004360 | ·003368 | 1·1 |
| 1·2 | ·018066 | ·015086 | ·012467 | ·010195 | ·008250 | ·006605 | ·005231 | ·004099 | ·003177 | 1·2 |
| 1·3 | ·016397 | ·013748 | ·011408 | ·009367 | ·007610 | ·006116 | ·004863 | ·003825 | ·002976 | 1·3 |
| 1·4 | ·014751 | ·012420 | ·010349 | ·008533 | ·006961 | ·005618 | ·004485 | ·003542 | ·002767 | 1·4 |
| 1·5 | ·013148 | ·011118 | ·009304 | ·007704 | ·006312 | ·005116 | ·004102 | ·003253 | ·002552 | 1·5 |
| 1·6 | ·011609 | ·009860 | ·008288 | ·006893 | ·005673 | ·004618 | ·003719 | ·002962 | ·002333 | 1·6 |
| 1·7 | ·010151 | ·008661 | ·007313 | ·006110 | ·005051 | ·004130 | ·003341 | ·002673 | ·002115 | 1·7 |
| 1·8 | ·008789 | ·007532 | ·006389 | ·005363 | ·004454 | ·003659 | ·002974 | ·002390 | ·001900 | 1·8 |
| 1·9 | ·007532 | ·006484 | ·005527 | ·004661 | ·003889 | ·003210 | ·002621 | ·002117 | ·001691 | 1·9 |
| 2·0 | ·006389 | ·005527 | ·004732 | ·004010 | ·003362 | ·002788 | ·002288 | ·001856 | ·001490 | 2·0 |
| 2·1 | ·005363 | ·004661 | ·004010 | ·003414 | ·002876 | ·002397 | ·001977 | ·001612 | ·001300 | 2·1 |
| 2·2 | ·004454 | ·003889 | ·003362 | ·002876 | ·002435 | ·002040 | ·001690 | ·001385 | ·001123 | 2·2 |
| 2·3 | ·003659 | ·003210 | ·002788 | ·002397 | ·002040 | ·001717 | ·001430 | ·001178 | ·000960 | 2·3 |
| 2·4 | ·002974 | ·002621 | ·002288 | ·001977 | ·001690 | ·001430 | ·001197 | ·000991 | ·000811 | 2·4 |
| 2·5 | ·002390 | ·002117 | ·001856 | ·001612 | ·001385 | ·001178 | ·000991 | ·000825 | ·000679 | 2·5 |
| 2·6 | ·001900 | ·001691 | ·001490 | ·001300 | ·001123 | ·000960 | ·000811 | ·000679 | ·000562 | 2·6 |

| $k$ | $r=\cdot60$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·352416 | ·331907 | ·310427 | ·288230 | ·265596 | ·242818 | ·220195 | ·198017 | ·176552 | $0\cdot0$ |
| $0\cdot1$ | ·331907 | ·313382 | ·293864 | ·273578 | ·252775 | ·231727 | ·210712 | ·190003 | ·169863 | $0\cdot1$ |
| $0\cdot2$ | ·310427 | ·293864 | ·276306 | ·257945 | ·239006 | ·219734 | ·200384 | ·181214 | ·162473 | $0\cdot2$ |
| $0\cdot3$ | ·288230 | ·273578 | ·257945 | ·241495 | ·224423 | ·206945 | ·189295 | ·171710 | ·154423 | $0\cdot3$ |
| $0\cdot4$ | ·265596 | ·252775 | ·239006 | ·224423 | ·209191 | ·193500 | ·177558 | ·161579 | ·145781 | $0\cdot4$ |
| $0\cdot5$ | ·242818 | ·231727 | ·219734 | ·206945 | ·193500 | ·179560 | ·165307 | ·150932 | ·136634 | $0\cdot5$ |
| $0\cdot6$ | ·220195 | ·210712 | ·200384 | ·189295 | ·177558 | ·165307 | ·152698 | ·139901 | ·127090 | $0\cdot6$ |
| $0\cdot7$ | ·198017 | ·190003 | ·181214 | ·171710 | ·161579 | ·150932 | ·139901 | ·128629 | ·117273 | $0\cdot7$ |
| $0\cdot8$ | ·176552 | ·169863 | ·162473 | ·154423 | ·145781 | ·136634 | ·127090 | ·117273 | ·107315 | $0\cdot8$ |
| $0\cdot9$ | ·156043 | ·150528 | ·144390 | ·137654 | ·130369 | ·122604 | ·114443 | ·105989 | ·097355 | $0\cdot9$ |
| $1\cdot0$ | ·136692 | ·132203 | ·127168 | ·121601 | ·115536 | ·109022 | ·102127 | ·094932 | ·087532 | $1\cdot0$ |
| $1\cdot1$ | ·118663 | ·115055 | ·110977 | ·106434 | ·101446 | ·096050 | ·090294 | ·084245 | ·077977 | $1\cdot1$ |
| $1\cdot2$ | ·102072 | ·099210 | ·095950 | ·092290 | ·088240 | ·083824 | ·079080 | ·074055 | ·068811 | $1\cdot2$ |
| $1\cdot3$ | ·086991 | ·084751 | ·082178 | ·079267 | ·076021 | ·072455 | ·068592 | ·064471 | ·060137 | $1\cdot3$ |
| $1\cdot4$ | ·073448 | ·071718 | ·069715 | ·067431 | ·064863 | ·062019 | ·058915 | ·055578 | ·052040 | $1\cdot4$ |
| $1\cdot5$ | ·061433 | ·060115 | ·058576 | ·056807 | ·054802 | ·052564 | ·050102 | ·047434 | ·044585 | $1\cdot5$ |
| $1\cdot6$ | ·050899 | ·049908 | ·048742 | ·047391 | ·045846 | ·044109 | ·042182 | ·040077 | ·037810 | $1\cdot6$ |
| $1\cdot7$ | ·041773 | ·041038 | ·040167 | ·039148 | ·037974 | ·036643 | ·035155 | ·033516 | ·031737 | $1\cdot7$ |
| $1\cdot8$ | ·033957 | ·033420 | ·032778 | ·032020 | ·031141 | ·030134 | ·029001 | ·027741 | ·026364 | $1\cdot8$ |
| $1\cdot9$ | ·027341 | ·026954 | ·026487 | ·025932 | ·025281 | ·024531 | ·023679 | ·022725 | ·021673 | $1\cdot9$ |
| $2\cdot0$ | ·021804 | ·021529 | ·021194 | ·020793 | ·020319 | ·019768 | ·019136 | ·018423 | ·017629 | $2\cdot0$ |
| $2\cdot1$ | ·017223 | ·017030 | ·016793 | ·016507 | ·016167 | ·015767 | ·015305 | ·014779 | ·014190 | $2\cdot1$ |
| $2\cdot2$ | ·013474 | ·013341 | ·013176 | ·012975 | ·012734 | ·012448 | ·012115 | ·011733 | ·011300 | $2\cdot2$ |
| $2\cdot3$ | ·010441 | ·010350 | ·010237 | ·010098 | ·009929 | ·009728 | ·009491 | ·009217 | ·008904 | $2\cdot3$ |
| $2\cdot4$ | ·008013 | ·007952 | ·007876 | ·007781 | ·007664 | ·007524 | ·007359 | ·007165 | ·006942 | $2\cdot4$ |
| $2\cdot5$ | ·006092 | ·006051 | ·006000 | ·005936 | ·005857 | ·005761 | ·005647 | ·005512 | ·005355 | $2\cdot5$ |
| $2\cdot6$ | ·004586 | ·004560 | ·004526 | ·004484 | ·004431 | ·004366 | ·004288 | ·004196 | ·004087 | $2\cdot6$ |

| $k$ | $r=\cdot60$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·156043 | ·136692 | ·118663 | ·102072 | ·086991 | ·073448 | ·061433 | ·050899 | ·041773 | $0\cdot0$ |
| $0\cdot1$ | ·150528 | ·132203 | ·115055 | ·099210 | ·084751 | ·071718 | ·060115 | ·049908 | ·041038 | $0\cdot1$ |
| $0\cdot2$ | ·144390 | ·127168 | ·110977 | ·095950 | ·082178 | ·069715 | ·058576 | ·048742 | ·040167 | $0\cdot2$ |
| $0\cdot3$ | ·137654 | ·121601 | ·106434 | ·092290 | ·079267 | ·067431 | ·056807 | ·047391 | ·039148 | $0\cdot3$ |
| $0\cdot4$ | ·130369 | ·115536 | ·101446 | ·088240 | ·076021 | ·064863 | ·054802 | ·045846 | ·037974 | $0\cdot4$ |
| $0\cdot5$ | ·122604 | ·109022 | ·096050 | ·083824 | ·072455 | ·062019 | ·052564 | ·044109 | ·036643 | $0\cdot5$ |
| $0\cdot6$ | ·114443 | ·102127 | ·090294 | ·079080 | ·068592 | ·058915 | ·050102 | ·042182 | ·035155 | $0\cdot6$ |
| $0\cdot7$ | ·105989 | ·094932 | ·084245 | ·074055 | ·064471 | ·055578 | ·047434 | ·040077 | ·033516 | $0\cdot7$ |
| $0\cdot8$ | ·097355 | ·087532 | ·077977 | ·068811 | ·060137 | ·052040 | ·044585 | ·037810 | ·031737 | $0\cdot8$ |
| $0\cdot9$ | ·088661 | ·080028 | ·071576 | ·063415 | ·055645 | ·048346 | ·041585 | ·035406 | ·029835 | $0\cdot9$ |
| $1\cdot0$ | ·080028 | ·072526 | ·065131 | ·057944 | ·051055 | ·044544 | ·038474 | ·032894 | ·027832 | $1\cdot0$ |
| $1\cdot1$ | ·071576 | ·065131 | ·058734 | ·052474 | ·046434 | ·040687 | ·035295 | ·030305 | ·025752 | $1\cdot1$ |
| $1\cdot2$ | ·063415 | ·057944 | ·052474 | ·047083 | ·041846 | ·036830 | ·032091 | ·027678 | ·023624 | $1\cdot2$ |
| $1\cdot3$ | ·055645 | ·051055 | ·046434 | ·041846 | ·037358 | ·033028 | ·028910 | ·025049 | ·021479 | $1\cdot3$ |
| $1\cdot4$ | ·048346 | ·044544 | ·040687 | ·036830 | ·033028 | ·029335 | ·025797 | ·022457 | ·019348 | $1\cdot4$ |
| $1\cdot5$ | ·041585 | ·038474 | ·035295 | ·032091 | ·028910 | ·025797 | ·022794 | ·019938 | ·017261 | $1\cdot5$ |
| $1\cdot6$ | ·035406 | ·032894 | ·030305 | ·027678 | ·025049 | ·022457 | ·019938 | ·017526 | ·015247 | $1\cdot6$ |
| $1\cdot7$ | ·029835 | ·027832 | ·025752 | ·023624 | ·021479 | ·019348 | ·017261 | ·015247 | ·013332 | $1\cdot7$ |
| $1\cdot8$ | ·024880 | ·023303 | ·021653 | ·019952 | ·018224 | ·016494 | ·014787 | ·013127 | ·011536 | $1\cdot8$ |
| $1\cdot9$ | ·020529 | ·019305 | ·018014 | ·016672 | ·015298 | ·013911 | ·012532 | ·011181 | ·009876 | $1\cdot9$ |
| $2\cdot0$ | ·016761 | ·015823 | ·014826 | ·013781 | ·012702 | ·011605 | ·010506 | ·009420 | ·008363 | $2\cdot0$ |
| $2\cdot1$ | ·013538 | ·012830 | ·012070 | ·011267 | ·010432 | ·009576 | ·008711 | ·007849 | ·007005 | $2\cdot1$ |
| $2\cdot2$ | ·010819 | ·010291 | ·009720 | ·009112 | ·008474 | ·007814 | ·007142 | ·006468 | ·005802 | $2\cdot2$ |
| $2\cdot3$ | ·008554 | ·008165 | ·007742 | ·007287 | ·006806 | ·006305 | ·005791 | ·005270 | ·004751 | $2\cdot3$ |
| $2\cdot4$ | ·006690 | ·006408 | ·006099 | ·005764 | ·005406 | ·005031 | ·004642 | ·004245 | ·003846 | $2\cdot4$ |
| $2\cdot5$ | ·005176 | ·004975 | ·004752 | ·004508 | ·004246 | ·003969 | ·003679 | ·003381 | ·003079 | $2\cdot5$ |
| $2\cdot6$ | ·003962 | ·003821 | ·003662 | ·003487 | ·003298 | ·003096 | ·002883 | ·002662 | ·002436 | $2\cdot6$ |

| $k$ | $r=\cdot60$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·033957 | ·027341 | ·021804 | ·017223 | ·013474 | ·010441 | ·008013 | ·006092 | ·004586 | 0·0 |
| 0·1 | ·033420 | ·026954 | ·021529 | ·017030 | ·013341 | ·010350 | ·007952 | ·006051 | ·004560 | 0·1 |
| 0·2 | ·032778 | ·026487 | ·021194 | ·016793 | ·013176 | ·010237 | ·007876 | ·006000 | ·004526 | 0·2 |
| 0·3 | ·032020 | ·025932 | ·020793 | ·016507 | ·012975 | ·010098 | ·007781 | ·005936 | ·004484 | 0·3 |
| 0·4 | ·031141 | ·025281 | ·020319 | ·016167 | ·012734 | ·009929 | ·007664 | ·005857 | ·004431 | 0·4 |
| 0·5 | ·030134 | ·024531 | ·019768 | ·015767 | ·012448 | ·009728 | ·007524 | ·005761 | ·004366 | 0·5 |
| 0·6 | ·029001 | ·023679 | ·019136 | ·015305 | ·012115 | ·009491 | ·007359 | ·005647 | ·004288 | 0·6 |
| 0·7 | ·027741 | ·022725 | ·018423 | ·014779 | ·011733 | ·009217 | ·007165 | ·005512 | ·004196 | 0·7 |
| 0·8 | ·026364 | ·021673 | ·017629 | ·014190 | ·011300 | ·008904 | ·006942 | ·005355 | ·004087 | 0·8 |
| 0·9 | ·024880 | ·020529 | ·016761 | ·013538 | ·010819 | ·008554 | ·006690 | ·005176 | ·003962 | 0·9 |
| 1·0 | ·023303 | ·019305 | ·015823 | ·012830 | ·010291 | ·008165 | ·006408 | ·004975 | ·003821 | 1·0 |
| 1·1 | ·021653 | ·018014 | ·014826 | ·012070 | ·009720 | ·007742 | ·006099 | ·004752 | ·003662 | 1·1 |
| 1·2 | ·019952 | ·016672 | ·013781 | ·011267 | ·009112 | ·007287 | ·005764 | ·004508 | ·003487 | 1·2 |
| 1·3 | ·018224 | ·015208 | ·012702 | ·010432 | ·008474 | ·006806 | ·005406 | ·004246 | ·003298 | 1·3 |
| 1·4 | ·016494 | ·013911 | ·011605 | ·009576 | ·007814 | ·006305 | ·005031 | ·003969 | ·003096 | 1·4 |
| 1·5 | ·014787 | ·012532 | ·010506 | ·008711 | ·007142 | ·005791 | ·004642 | ·003679 | ·002883 | 1·5 |
| 1·6 | ·013127 | ·011181 | ·009420 | ·007849 | ·006468 | ·005270 | ·004245 | ·003381 | ·002662 | 1·6 |
| 1·7 | ·011536 | ·009876 | ·008363 | ·007005 | ·005802 | ·004751 | ·003846 | ·003079 | ·002436 | 1·7 |
| 1·8 | ·010033 | ·008634 | ·007350 | ·006188 | ·005153 | ·004242 | ·003452 | ·002778 | ·002209 | 1·8 |
| 1·9 | ·008634 | ·007469 | ·006392 | ·005411 | ·004529 | ·003749 | ·003068 | ·002481 | ·001984 | 1·9 |
| 2·0 | ·007350 | ·006392 | ·005500 | ·004681 | ·003940 | ·003279 | ·002698 | ·002194 | ·001764 | 2·0 |
| 2·1 | ·006188 | ·005411 | ·004681 | ·004006 | ·003390 | ·002837 | ·002348 | ·001920 | ·001552 | 2·1 |
| 2·2 | ·005153 | ·004529 | ·003940 | ·003390 | ·002886 | ·002429 | ·002021 | ·001662 | ·001351 | 2·2 |
| 2·3 | ·004242 | ·003749 | ·003279 | ·002837 | ·002429 | ·002056 | ·001720 | ·001423 | ·001164 | 2·3 |
| 2·4 | ·003452 | ·003068 | ·002698 | ·002348 | ·002021 | ·001720 | ·001448 | ·001205 | ·000991 | 2·4 |
| 2·5 | ·002778 | ·002481 | ·002194 | ·001920 | ·001662 | ·001423 | ·001205 | ·001009 | ·000835 | 2·5 |
| 2·6 | ·002209 | ·001984 | ·001764 | ·001552 | ·001351 | ·001164 | ·000991 | ·000835 | ·000695 | 2·6 |

| $k$ | $r=\cdot65$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| 0·0 | ·362615 | ·342023 | ·320297 | ·297702 | ·274538 | ·251123 | ·227783 | ·204836 | ·182581 | 0·0 |
| 0·1 | ·342023 | ·323519 | ·303856 | ·283267 | ·262017 | ·240399 | ·218716 | ·197271 | ·176354 | 0·1 |
| 0·2 | ·320297 | ·303856 | ·286256 | ·267694 | ·248401 | ·228640 | ·208690 | ·188833 | ·169349 | 0·2 |
| 0·3 | ·297702 | ·283267 | ·267694 | ·251145 | ·233818 | ·215944 | ·197774 | ·179568 | ·161588 | 0·3 |
| 0·4 | ·274538 | ·262017 | ·248401 | ·233818 | ·218433 | ·202444 | ·186072 | ·169552 | ·153125 | 0·4 |
| 0·5 | ·251123 | ·240399 | ·228640 | ·215944 | ·202444 | ·188305 | ·173717 | ·158889 | ·144039 | 0·5 |
| 0·6 | ·227783 | ·218716 | ·208690 | ·197774 | ·186072 | ·173717 | ·160870 | ·147712 | ·134436 | 0·6 |
| 0·7 | ·204836 | ·197271 | ·188833 | ·179568 | ·169552 | ·158889 | ·147712 | ·136173 | ·124440 | 0·7 |
| 0·8 | ·182581 | ·176354 | ·169349 | ·161588 | ·153125 | ·144039 | ·134436 | ·124440 | ·114194 | 0·8 |
| 0·9 | ·161285 | ·156232 | ·150493 | ·144080 | ·137024 | ·129383 | ·121237 | ·112687 | ·103850 | 0·9 |
| 1·0 | ·141177 | ·137133 | ·132498 | ·127270 | ·121468 | ·115127 | ·108308 | ·101089 | ·093564 | 1·0 |
| 1·1 | ·122437 | ·119247 | ·115556 | ·111355 | ·106648 | ·101459 | ·095827 | ·089813 | ·083489 | 1·1 |
| 1·2 | ·105196 | ·102716 | ·099819 | ·096490 | ·092726 | ·088538 | ·083951 | ·079008 | ·073764 | 1·2 |
| 1·3 | ·089536 | ·087635 | ·085395 | ·082795 | ·079828 | ·076495 | ·072812 | ·068805 | ·064516 | 1·3 |
| 1·4 | ·075487 | ·074053 | ·072345 | ·070345 | ·068040 | ·065426 | ·062510 | ·059308 | ·055849 | 1·4 |
| 1·5 | ·063040 | ·061974 | ·060692 | ·059176 | ·057411 | ·055391 | ·053115 | ·050594 | ·047843 | 1·5 |
| 1·6 | ·052146 | ·051365 | ·050417 | ·049284 | ·047954 | ·046415 | ·044666 | ·042708 | ·040552 | 1·6 |
| 1·7 | ·042723 | ·042161 | ·041470 | ·040637 | ·039649 | ·038495 | ·037169 | ·035672 | ·034007 | 1·7 |
| 1·8 | ·034671 | ·034271 | ·033776 | ·033173 | ·032450 | ·031597 | ·030608 | ·029479 | ·028213 | 1·8 |
| 1·9 | ·027868 | ·027589 | ·027239 | ·026809 | ·026288 | ·025668 | ·024941 | ·024103 | ·023154 | 1·9 |
| 2·0 | ·022187 | ·021995 | ·021752 | ·021450 | ·021080 | ·020636 | ·020110 | ·019497 | ·018797 | 2·0 |
| 2·1 | ·017497 | ·017366 | ·017200 | ·016991 | ·016733 | ·016420 | ·016045 | ·015604 | ·015094 | 2·1 |
| 2·2 | ·013667 | ·013580 | ·013468 | ·013326 | ·013149 | ·012931 | ·012668 | ·012355 | ·011991 | 2·2 |
| 2·3 | ·010574 | ·010517 | ·010443 | ·010348 | ·010228 | ·010079 | ·009897 | ·009679 | ·009422 | 2·3 |
| 2·4 | ·008104 | ·008067 | ·008019 | ·007956 | ·007876 | ·007776 | ·007653 | ·007503 | ·007324 | 2·4 |
| 2·5 | ·006152 | ·006129 | ·006098 | ·006057 | ·006005 | ·005939 | ·005856 | ·005755 | ·005633 | 2·5 |
| 2·6 | ·004626 | ·004612 | ·004592 | ·004566 | ·004533 | ·004489 | ·004435 | ·004368 | ·004286 | 2·6 |

| $k$ | $r=\cdot65$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·161285 | ·141177 | ·122437 | ·105196 | ·089536 | ·075487 | ·063040 | ·052146 | ·042723 | $0\cdot0$ |
| $0\cdot1$ | ·156232 | ·137133 | ·119247 | ·102716 | ·087635 | ·074053 | ·061974 | ·051365 | ·042161 | $0\cdot1$ |
| $0\cdot2$ | ·150493 | ·132498 | ·115556 | ·099819 | ·085395 | ·072345 | ·060692 | ·050417 | ·041470 | $0\cdot2$ |
| $0\cdot3$ | ·144080 | ·127270 | ·111355 | ·096490 | ·082795 | ·070345 | ·059176 | ·049284 | ·040637 | $0\cdot3$ |
| $0\cdot4$ | ·137024 | ·121468 | ·106648 | ·092726 | ·079828 | ·068040 | ·057411 | ·047954 | ·039649 | $0\cdot4$ |
| $0\cdot5$ | ·129383 | ·115127 | ·101459 | ·088538 | ·076495 | ·065426 | ·055391 | ·046415 | ·038495 | $0\cdot5$ |
| $0\cdot6$ | ·121237 | ·108308 | ·095827 | ·083951 | ·072812 | ·062510 | ·053115 | ·044666 | ·037169 | $0\cdot6$ |
| $0\cdot7$ | ·112687 | ·101089 | ·089813 | ·079008 | ·068805 | ·059308 | ·050594 | ·042708 | ·035672 | $0\cdot7$ |
| $0\cdot8$ | ·103850 | ·093564 | ·083489 | ·073764 | ·064516 | ·055849 | ·047843 | ·040552 | ·034007 | $0\cdot8$ |
| $0\cdot9$ | ·094856 | ·085842 | ·076943 | ·068288 | ·059997 | ·052171 | ·044891 | ·038217 | ·032187 | $0\cdot9$ |
| $1\cdot0$ | ·085842 | ·078038 | ·070272 | ·062660 | ·055311 | ·048323 | ·041774 | ·035728 | ·030227 | $1\cdot0$ |
| $1\cdot1$ | ·076943 | ·070272 | ·063578 | ·056964 | ·050527 | ·044359 | ·038534 | ·033117 | ·028153 | $1\cdot1$ |
| $1\cdot2$ | ·068288 | ·062660 | ·056964 | ·051288 | ·045720 | ·040340 | ·035220 | ·030422 | ·025991 | $1\cdot2$ |
| $1\cdot3$ | ·059997 | ·055311 | ·050527 | ·045720 | ·040963 | ·036329 | ·031883 | ·027683 | ·023775 | $1\cdot3$ |
| $1\cdot4$ | ·052171 | ·048323 | ·044359 | ·040340 | ·036329 | ·032388 | ·028576 | ·024944 | ·021538 | $1\cdot4$ |
| $1\cdot5$ | ·044891 | ·041774 | ·038534 | ·035220 | ·031883 | ·028576 | ·025349 | ·022249 | ·019316 | $1\cdot5$ |
| $1\cdot6$ | ·038217 | ·035728 | ·033117 | ·030422 | ·027683 | ·024944 | ·022249 | ·019636 | ·017144 | $1\cdot6$ |
| $1\cdot7$ | ·032187 | ·030227 | ·028153 | ·025991 | ·023775 | ·021538 | ·019316 | ·017144 | ·015055 | $1\cdot7$ |
| $1\cdot8$ | ·026814 | ·025294 | ·023670 | ·021961 | ·020192 | ·018391 | ·016585 | ·014804 | ·013075 | $1\cdot8$ |
| $1\cdot9$ | ·022095 | ·020933 | ·019680 | ·018348 | ·016957 | ·015527 | ·014080 | ·012639 | ·011228 | $1\cdot9$ |
| $2\cdot0$ | ·018007 | ·017133 | ·016179 | ·015157 | ·014079 | ·012960 | ·011817 | ·010668 | ·009532 | $2\cdot0$ |
| $2\cdot1$ | ·014515 | ·013866 | ·013152 | ·012379 | ·011556 | ·010693 | ·009803 | ·008900 | ·007999 | $2\cdot1$ |
| $2\cdot2$ | ·011571 | ·011097 | ·010571 | ·009995 | ·009375 | ·008720 | ·008037 | ·007338 | ·006633 | $2\cdot2$ |
| $2\cdot3$ | ·009124 | ·008783 | ·008400 | ·007978 | ·007519 | ·007028 | ·006512 | ·005978 | ·005436 | $2\cdot3$ |
| $2\cdot4$ | ·007115 | ·006874 | ·006600 | ·006295 | ·005960 | ·005598 | ·005214 | ·004813 | ·004401 | $2\cdot4$ |
| $2\cdot5$ | ·005488 | ·005320 | ·005127 | ·004910 | ·004669 | ·004407 | ·004125 | ·003828 | ·003520 | $2\cdot5$ |
| $2\cdot6$ | ·004187 | ·004072 | ·003938 | ·003786 | ·003616 | ·003428 | ·003225 | ·003008 | ·002781 | $2\cdot6$ |

| $k$ | $r=\cdot65$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·034671 | ·027868 | ·022187 | ·017497 | ·013667 | ·010574 | ·008104 | ·006152 | ·004626 | $0\cdot0$ |
| $0\cdot1$ | ·034271 | ·027589 | ·021995 | ·017366 | ·013580 | ·010517 | ·008067 | ·006129 | ·004612 | $0\cdot1$ |
| $0\cdot2$ | ·033776 | ·027239 | ·021752 | ·017200 | ·013468 | ·010443 | ·008019 | ·006098 | ·004592 | $0\cdot2$ |
| $0\cdot3$ | ·033173 | ·026809 | ·021450 | ·016991 | ·013326 | ·010348 | ·007956 | ·006057 | ·004566 | $0\cdot3$ |
| $0\cdot4$ | ·032450 | ·026288 | ·021080 | ·016733 | ·013149 | ·010228 | ·007876 | ·006005 | ·004533 | $0\cdot4$ |
| $0\cdot5$ | ·031597 | ·025668 | ·020636 | ·016420 | ·012931 | ·010079 | ·007776 | ·005939 | ·004489 | $0\cdot5$ |
| $0\cdot6$ | ·030608 | ·024941 | ·020110 | ·016045 | ·012668 | ·009897 | ·007653 | ·005856 | ·004435 | $0\cdot6$ |
| $0\cdot7$ | ·029479 | ·024103 | ·019497 | ·015604 | ·012355 | ·009679 | ·007503 | ·005755 | ·004368 | $0\cdot7$ |
| $0\cdot8$ | ·028213 | ·023154 | ·018797 | ·015094 | ·011991 | ·009422 | ·007324 | ·005633 | ·004286 | $0\cdot8$ |
| $0\cdot9$ | ·026814 | ·022095 | ·018007 | ·014515 | ·011571 | ·009124 | ·007115 | ·005488 | ·004187 | $0\cdot9$ |
| $1\cdot0$ | ·025294 | ·020933 | ·017133 | ·013866 | ·011097 | ·008783 | ·006874 | ·005320 | ·004072 | $1\cdot0$ |
| $1\cdot1$ | ·023670 | ·019680 | ·016179 | ·013152 | ·010571 | ·008400 | ·006600 | ·005127 | ·003938 | $1\cdot1$ |
| $1\cdot2$ | ·021961 | ·018348 | ·015157 | ·012379 | ·009995 | ·007978 | ·006295 | ·004910 | ·003786 | $1\cdot2$ |
| $1\cdot3$ | ·020192 | ·016957 | ·014079 | ·011556 | ·009375 | ·007519 | ·005960 | ·004669 | ·003616 | $1\cdot3$ |
| $1\cdot4$ | ·018391 | ·015527 | ·012960 | ·010693 | ·008720 | ·007028 | ·005598 | ·004407 | ·003428 | $1\cdot4$ |
| $1\cdot5$ | ·016585 | ·014080 | ·011817 | ·009803 | ·008037 | ·006512 | ·005214 | ·004125 | ·003225 | $1\cdot5$ |
| $1\cdot6$ | ·014804 | ·012639 | ·010668 | ·008900 | ·007338 | ·005978 | ·004813 | ·003828 | ·003008 | $1\cdot6$ |
| $1\cdot7$ | ·013075 | ·011228 | ·009532 | ·007999 | ·006633 | ·005436 | ·004401 | ·003520 | ·002781 | $1\cdot7$ |
| $1\cdot8$ | ·011422 | ·009867 | ·008427 | ·007114 | ·005935 | ·004892 | ·003984 | ·003205 | ·002547 | $1\cdot8$ |
| $1\cdot9$ | ·009867 | ·008575 | ·007368 | ·006258 | ·005253 | ·004357 | ·003570 | ·002889 | ·002310 | $1\cdot9$ |
| $2\cdot0$ | ·008427 | ·007368 | ·006370 | ·005444 | ·004598 | ·003838 | ·003164 | ·002577 | ·002073 | $2\cdot0$ |
| $2\cdot1$ | ·007114 | ·006258 | ·005444 | ·004682 | ·003980 | ·003343 | ·002774 | ·002274 | ·001841 | $2\cdot1$ |
| $2\cdot2$ | ·005935 | ·005253 | ·004598 | ·003980 | ·003405 | ·002878 | ·002404 | ·001983 | ·001616 | $2\cdot2$ |
| $2\cdot3$ | ·004892 | ·004357 | ·003838 | ·003343 | ·002878 | ·002449 | ·002059 | ·001710 | ·001403 | $2\cdot3$ |
| $2\cdot4$ | ·003984 | ·003570 | ·003164 | ·002774 | ·002404 | ·002059 | ·001743 | ·001457 | ·001204 | $2\cdot4$ |
| $2\cdot5$ | ·003205 | ·002889 | ·002577 | ·002274 | ·001983 | ·001710 | ·001457 | ·001227 | ·001020 | $2\cdot5$ |
| $2\cdot6$ | ·002547 | ·002310 | ·002073 | ·001841 | ·001616 | ·001403 | ·001204 | ·001020 | ·000854 | $2\cdot6$ |

| $k$ | $r = \cdot 70$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 0 \cdot 0$ | $h = 0 \cdot 1$ | $h = 0 \cdot 2$ | $h = 0 \cdot 3$ | $h = 0 \cdot 4$ | $h = 0 \cdot 5$ | $h = 0 \cdot 6$ | $h = 0 \cdot 7$ | $h = 0 \cdot 8$ | |
| 0·0 | ·373407 | ·352717 | ·330699 | ·307637 | ·283853 | ·259697 | ·235531 | ·211712 | ·188570 | 0·0 |
| 0·1 | ·352717 | ·334247 | ·314423 | ·293485 | ·271717 | ·249439 | ·226988 | ·204702 | ·182909 | 0·1 |
| 0·2 | ·330699 | ·314423 | ·296795 | ·278012 | ·258319 | ·237999 | ·217360 | ·196720 | ·176391 | 0·2 |
| 0·3 | ·307637 | ·293485 | ·278012 | ·261373 | ·243773 | ·225456 | ·206696 | ·187786 | ·169017 | 0·3 |
| 0·4 | ·283853 | ·271717 | ·258319 | ·243773 | ·228243 | ·211935 | ·195086 | ·177957 | ·160820 | 0·4 |
| 0·5 | ·259697 | ·249439 | ·237999 | ·225456 | ·211935 | ·197602 | ·182658 | ·167331 | ·151864 | 0·5 |
| 0·6 | ·235531 | ·226988 | ·217360 | ·206696 | ·195086 | ·182658 | ·169576 | ·156035 | ·142246 | 0·6 |
| 0·7 | ·211712 | ·204702 | ·196720 | ·187786 | ·177957 | ·167331 | ·156035 | ·144229 | ·132095 | 0·7 |
| 0·8 | ·188570 | ·182909 | ·176391 | ·169017 | ·160820 | ·151864 | ·142246 | ·132095 | ·121559 | 0·8 |
| 0·9 | ·166407 | ·161907 | ·156668 | ·150675 | ·143940 | ·136503 | ·128434 | ·119829 | ·110808 | 0·9 |
| 1·0 | ·145478 | ·141958 | ·137812 | ·133018 | ·127570 | ·121489 | ·114819 | ·107630 | ·100017 | 1·0 |
| 1·1 | ·125983 | ·123274 | ·120047 | ·116272 | ·111934 | ·107038 | ·101609 | ·095695 | ·089364 | 1·1 |
| 1·2 | ·108067 | ·106016 | ·103546 | ·100621 | ·097223 | ·093343 | ·088993 | ·084201 | ·079016 | 1·2 |
| 1·3 | ·091817 | ·090291 | ·088430 | ·086203 | ·083583 | ·080558 | ·077127 | ·073306 | ·069125 | 1·3 |
| 1·4 | ·077268 | ·076151 | ·074774 | ·073104 | ·071118 | ·068798 | ·066137 | ·063138 | ·059820 | 1·4 |
| 1·5 | ·064404 | ·063601 | ·062599 | ·061369 | ·059889 | ·058139 | ·056108 | ·053793 | ·051201 | 1·5 |
| 1·6 | ·053172 | ·052604 | ·051887 | ·050997 | ·049912 | ·048614 | ·047090 | ·045332 | ·043341 | 1·6 |
| 1·7 | ·043481 | ·043088 | ·042583 | ·041950 | ·041169 | ·040222 | ·039098 | ·037785 | ·036281 | 1·7 |
| 1·8 | ·035221 | ·034952 | ·034604 | ·034161 | ·033608 | ·032930 | ·032114 | ·031150 | ·030033 | 1·8 |
| 1·9 | ·028260 | ·028080 | ·027843 | ·027539 | ·027154 | ·026677 | ·026095 | ·025400 | ·024584 | 1·9 |
| 2·0 | ·022461 | ·022343 | ·022185 | ·021980 | ·021717 | ·021386 | ·020979 | ·020486 | ·019901 | 2·0 |
| 2·1 | ·017685 | ·017608 | ·017505 | ·017369 | ·017192 | ·016968 | ·016687 | ·016344 | ·015931 | 2·1 |
| 2·2 | ·013793 | ·013745 | ·013679 | ·013590 | ·013474 | ·013324 | ·013134 | ·012899 | ·012613 | 2·2 |
| 2·3 | ·010658 | ·010628 | ·010586 | ·010529 | ·010454 | ·010356 | ·010230 | ·010072 | ·009877 | 2·3 |
| 2·4 | ·008159 | ·008140 | ·008114 | ·008079 | ·008031 | ·007967 | ·007885 | ·007781 | ·007650 | 2·4 |
| 2·5 | ·006187 | ·006176 | ·006160 | ·006138 | ·006108 | ·006068 | ·006016 | ·005948 | ·005862 | 2·5 |
| 2·6 | ·004648 | ·004642 | ·004632 | ·004619 | ·004601 | ·004576 | ·004543 | ·004500 | ·004444 | 2·6 |

| $k$ | $r = \cdot 70$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 0 \cdot 9$ | $h = 1 \cdot 0$ | $h = 1 \cdot 1$ | $h = 1 \cdot 2$ | $h = 1 \cdot 3$ | $h = 1 \cdot 4$ | $h = 1 \cdot 5$ | $h = 1 \cdot 6$ | $h = 1 \cdot 7$ | |
| 0·0 | ·166407 | ·145478 | ·125983 | ·108067 | ·091817 | ·077268 | ·064404 | ·053172 | ·043481 | 0·0 |
| 0·1 | ·161907 | ·141958 | ·123274 | ·106016 | ·090291 | ·076151 | ·063601 | ·052604 | ·043088 | 0·1 |
| 0·2 | ·156668 | ·137812 | ·120047 | ·103546 | ·088430 | ·074774 | ·062599 | ·051887 | ·042583 | 0·2 |
| 0·3 | ·150675 | ·133018 | ·116272 | ·100621 | ·086203 | ·073104 | ·061369 | ·050997 | ·041950 | 0·3 |
| 0·4 | ·143940 | ·127570 | ·111934 | ·097223 | ·083583 | ·071118 | ·059889 | ·049912 | ·041169 | 0·4 |
| 0·5 | ·136503 | ·121489 | ·107038 | ·093343 | ·080558 | ·068798 | ·058139 | ·048614 | ·040222 | 0·5 |
| 0·6 | ·128434 | ·114819 | ·101609 | ·088993 | ·077127 | ·066137 | ·056108 | ·047090 | ·039098 | 0·6 |
| 0·7 | ·119829 | ·107630 | ·095695 | ·084201 | ·073306 | ·063138 | ·053793 | ·045332 | ·037785 | 0·7 |
| 0·8 | ·110808 | ·100017 | ·089364 | ·079016 | ·069125 | ·059820 | ·051201 | ·043341 | ·036281 | 0·8 |
| 0·9 | ·101511 | ·092091 | ·082703 | ·073502 | ·064630 | ·056212 | ·048352 | ·041127 | ·034589 | 0·9 |
| 1·0 | ·092091 | ·083979 | ·075818 | ·067742 | ·059884 | ·052361 | ·045276 | ·038709 | ·032719 | 1·0 |
| 1·1 | ·082703 | ·075818 | ·068819 | ·061827 | ·054958 | ·048320 | ·042013 | ·036115 | ·030690 | 1·1 |
| 1·2 | ·073502 | ·067742 | ·061827 | ·055857 | ·049934 | ·044156 | ·038612 | ·033382 | ·028528 | 1·2 |
| 1·3 | ·064630 | ·059884 | ·054958 | ·049934 | ·044899 | ·039937 | ·035131 | ·030553 | ·026265 | 1·3 |
| 1·4 | ·056212 | ·052361 | ·048320 | ·044156 | ·039937 | ·035738 | ·031629 | ·027676 | ·023937 | 1·4 |
| 1·5 | ·048352 | ·045276 | ·042013 | ·038612 | ·035131 | ·031629 | ·028166 | ·024801 | ·021587 | 1·5 |
| 1·6 | ·041127 | ·038709 | ·036115 | ·033382 | ·030553 | ·027676 | ·024801 | ·021978 | ·019253 | 1·6 |
| 1·7 | ·034589 | ·032719 | ·030690 | ·028528 | ·026265 | ·023937 | ·021587 | ·019253 | ·016977 | 1·7 |
| 1·8 | ·028761 | ·027339 | ·025778 | ·024095 | ·022313 | ·020460 | ·018568 | ·016668 | ·014796 | 1·8 |
| 1·9 | ·023644 | ·022581 | ·021399 | ·020111 | ·018731 | ·017279 | ·015779 | ·014258 | ·012741 | 1·9 |
| 2·0 | ·019217 | ·018435 | ·017556 | ·016586 | ·015535 | ·014416 | ·013247 | ·012047 | ·010838 | 2·0 |
| 2·1 | ·015443 | ·014878 | ·014235 | ·013517 | ·012729 | ·011881 | ·010984 | ·010054 | ·009105 | 2·1 |
| 2·2 | ·012271 | ·011869 | ·011407 | ·010884 | ·010304 | ·009672 | ·008996 | ·008285 | ·007553 | 2·2 |
| 2·3 | ·009641 | ·009361 | ·009034 | ·008660 | ·008240 | ·007777 | ·007275 | ·006742 | ·006186 | 2·3 |
| 2·4 | ·007491 | ·007298 | ·007072 | ·006809 | ·006510 | ·006176 | ·005811 | ·005418 | ·005002 | 2·4 |
| 2·5 | ·005756 | ·005626 | ·005472 | ·005290 | ·005081 | ·004845 | ·004583 | ·004298 | ·003993 | 2·5 |
| 2·6 | ·004375 | ·004289 | ·004185 | ·004062 | ·003919 | ·003755 | ·003570 | ·003367 | ·003147 | 2·6 |

| $k$ | $r=\cdot70$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·035221 | ·028260 | ·022461 | ·017685 | ·013793 | ·010658 | ·008159 | ·006187 | ·004648 | $0\cdot0$ |
| $0\cdot1$ | ·034952 | ·028080 | ·022343 | ·017608 | ·013745 | ·010628 | ·008140 | ·006176 | ·004642 | $0\cdot1$ |
| $0\cdot2$ | ·034604 | ·027843 | ·022185 | ·017505 | ·013679 | ·010586 | ·008114 | ·006160 | ·004632 | $0\cdot2$ |
| $0\cdot3$ | ·034161 | ·027539 | ·021980 | ·017369 | ·013590 | ·010529 | ·008079 | ·006138 | ·004619 | $0\cdot3$ |
| $0\cdot4$ | ·033608 | ·027154 | ·021717 | ·017192 | ·013474 | ·010454 | ·008031 | ·006108 | ·004601 | $0\cdot4$ |
| $0\cdot5$ | ·032930 | ·026677 | ·021386 | ·016968 | ·013324 | ·010356 | ·007967 | ·006068 | ·004576 | $0\cdot5$ |
| $0\cdot6$ | ·032114 | ·026095 | ·020979 | ·016687 | ·013134 | ·010230 | ·007885 | ·006016 | ·004543 | $0\cdot6$ |
| $0\cdot7$ | ·031150 | ·025400 | ·020486 | ·016344 | ·012899 | ·010072 | ·007781 | ·005948 | ·004500 | $0\cdot7$ |
| $0\cdot8$ | ·030033 | ·024584 | ·019901 | ·015931 | ·012613 | ·009877 | ·007650 | ·005862 | ·004444 | $0\cdot8$ |
| $0\cdot9$ | ·028761 | ·023644 | ·019217 | ·015443 | ·012271 | ·009641 | ·007491 | ·005756 | ·004375 | $0\cdot9$ |
| $1\cdot0$ | ·027339 | ·022581 | ·018435 | ·014878 | ·011869 | ·009361 | ·007298 | ·005626 | ·004289 | $1\cdot0$ |
| $1\cdot1$ | ·025778 | ·021399 | ·017556 | ·014235 | ·011407 | ·009034 | ·007072 | ·005472 | ·004185 | $1\cdot1$ |
| $1\cdot2$ | ·024095 | ·020111 | ·016586 | ·013517 | ·010884 | ·008660 | ·006809 | ·005290 | ·004062 | $1\cdot2$ |
| $1\cdot3$ | ·022313 | ·018731 | ·015535 | ·012729 | ·010304 | ·008240 | ·006510 | ·005081 | ·003919 | $1\cdot3$ |
| $1\cdot4$ | ·020460 | ·017279 | ·014416 | ·011881 | ·009672 | ·007777 | ·006176 | ·004845 | ·003755 | $1\cdot4$ |
| $1\cdot5$ | ·018568 | ·015779 | ·013247 | ·010984 | ·008996 | ·007275 | ·005811 | ·004583 | ·003570 | $1\cdot5$ |
| $1\cdot6$ | ·016668 | ·014258 | ·012047 | ·010054 | ·008285 | ·006742 | ·005418 | ·004298 | ·003367 | $1\cdot6$ |
| $1\cdot7$ | ·014796 | ·012741 | ·010838 | ·009105 | ·007553 | ·006186 | ·005002 | ·003993 | ·003147 | $1\cdot7$ |
| $1\cdot8$ | ·012982 | ·011256 | ·009640 | ·008154 | ·006811 | ·005616 | ·004572 | ·003674 | ·002914 | $1\cdot8$ |
| $1\cdot9$ | ·011256 | ·009826 | ·008475 | ·007219 | ·006072 | ·005042 | ·004133 | ·003344 | ·002670 | $1\cdot9$ |
| $2\cdot0$ | ·009640 | ·008475 | ·007362 | ·006316 | ·005351 | ·004475 | ·003695 | ·003010 | ·002421 | $2\cdot0$ |
| $2\cdot1$ | ·008154 | ·007219 | ·006316 | ·005458 | ·004658 | ·003925 | ·003264 | ·002679 | ·002170 | $2\cdot1$ |
| $2\cdot2$ | ·006811 | ·006072 | ·005351 | ·004658 | ·004005 | ·003400 | ·002849 | ·002356 | ·001923 | $2\cdot2$ |
| $2\cdot3$ | ·005616 | ·005042 | ·004475 | ·003925 | ·003400 | ·002908 | ·002456 | ·002046 | ·001683 | $2\cdot3$ |
| $2\cdot4$ | ·004572 | ·004133 | ·003695 | ·003264 | ·002849 | ·002456 | ·002090 | ·001755 | ·001455 | $2\cdot4$ |
| $2\cdot5$ | ·003674 | ·003344 | ·003010 | ·002679 | ·002356 | ·002046 | ·001755 | ·001486 | ·001241 | $2\cdot5$ |
| $2\cdot6$ | ·002914 | ·002670 | ·002421 | ·002170 | ·001923 | ·001683 | ·001455 | ·001241 | ·001045 | $2\cdot6$ |

| $k$ | $r=\cdot75$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·384973 | ·364160 | ·341785 | ·318152 | ·293618 | ·268576 | ·243436 | ·218600 | ·194450 | $0\cdot0$ |
| $0\cdot1$ | ·364160 | ·345745 | ·325735 | ·304380 | ·281991 | ·258925 | ·235563 | ·212292 | ·189486 | $0\cdot1$ |
| $0\cdot2$ | ·341785 | ·325735 | ·308095 | ·289064 | ·268902 | ·247922 | ·226469 | ·204906 | ·183595 | $0\cdot2$ |
| $0\cdot3$ | ·318152 | ·304380 | ·289064 | ·272350 | ·254447 | ·235619 | ·216170 | ·196431 | ·176743 | $0\cdot3$ |
| $0\cdot4$ | ·293618 | ·281991 | ·268902 | ·254447 | ·238784 | ·222126 | ·204732 | ·186897 | ·168931 | $0\cdot4$ |
| $0\cdot5$ | ·268576 | ·258925 | ·247922 | ·235619 | ·222126 | ·207607 | ·192274 | ·176381 | ·160203 | $0\cdot5$ |
| $0\cdot6$ | ·243436 | ·235563 | ·226469 | ·216170 | ·204732 | ·192274 | ·178963 | ·165005 | ·150640 | $0\cdot6$ |
| $0\cdot7$ | ·218600 | ·212292 | ·204906 | ·196431 | ·186897 | ·176381 | ·165005 | ·152934 | ·140366 | $0\cdot7$ |
| $0\cdot8$ | ·194450 | ·189486 | ·183595 | ·176743 | ·168931 | ·160203 | ·150640 | ·140366 | ·129540 | $0\cdot8$ |
| $0\cdot9$ | ·171321 | ·167488 | ·162875 | ·157434 | ·151146 | ·144026 | ·136123 | ·127523 | ·118347 | $0\cdot9$ |
| $1\cdot0$ | ·149499 | ·146595 | ·143049 | ·138808 | ·133838 | ·128133 | ·121715 | ·114639 | ·106989 | $1\cdot0$ |
| $1\cdot1$ | ·129205 | ·127047 | ·124373 | ·121129 | ·117273 | ·112784 | ·107665 | ·101944 | ·095677 | $1\cdot1$ |
| $1\cdot2$ | ·110594 | ·109022 | ·107044 | ·104610 | ·101674 | ·098208 | ·094200 | ·089657 | ·084613 | $1\cdot2$ |
| $1\cdot3$ | ·093759 | ·092635 | ·091201 | ·089410 | ·087218 | ·084592 | ·081511 | ·077970 | ·073983 | $1\cdot3$ |
| $1\cdot4$ | ·078729 | ·077942 | ·076922 | ·075630 | ·074024 | ·072073 | ·069750 | ·067041 | ·063949 | $1\cdot4$ |
| $1\cdot5$ | ·065481 | ·064941 | ·064231 | ·063316 | ·062164 | ·060742 | ·059024 | ·056992 | ·054637 | $1\cdot5$ |
| $1\cdot6$ | ·053949 | ·053586 | ·053101 | ·052467 | ·051656 | ·050640 | ·049395 | ·047899 | ·046141 | $1\cdot6$ |
| $1\cdot7$ | ·044031 | ·043792 | ·043467 | ·043037 | ·042478 | ·041766 | ·040881 | ·039802 | ·038514 | $1\cdot7$ |
| $1\cdot8$ | ·035601 | ·035447 | ·035234 | ·034948 | ·034570 | ·034082 | ·033465 | ·032702 | ·031778 | $1\cdot8$ |
| $1\cdot9$ | ·028517 | ·028420 | ·028284 | ·028097 | ·027847 | ·027519 | ·027098 | ·026569 | ·025919 | $1\cdot9$ |
| $2\cdot0$ | ·022632 | ·022572 | ·022486 | ·022367 | ·022205 | ·021989 | ·021708 | ·021348 | ·020900 | $2\cdot0$ |
| $2\cdot1$ | ·017796 | ·017760 | ·017707 | ·017633 | ·017530 | ·017390 | ·017206 | ·016967 | ·016664 | $2\cdot1$ |
| $2\cdot2$ | ·013865 | ·013843 | ·013811 | ·013766 | ·013702 | ·013614 | ·013495 | ·013340 | ·013139 | $2\cdot2$ |
| $2\cdot3$ | ·010702 | ·010690 | ·010671 | ·010644 | ·010605 | ·010551 | ·010476 | ·010377 | ·010247 | $2\cdot3$ |
| $2\cdot4$ | ·008186 | ·008179 | ·008168 | ·008152 | ·008129 | ·008096 | ·008050 | ·007988 | ·007905 | $2\cdot4$ |
| $2\cdot5$ | ·006203 | ·006199 | ·006193 | ·006184 | ·006171 | ·006151 | ·006124 | ·006086 | ·006034 | $2\cdot5$ |
| $2\cdot6$ | ·004658 | ·004656 | ·004652 | ·004647 | ·004639 | ·004628 | ·004612 | ·004589 | ·004558 | $2\cdot6$ |

| $k$ | $r=\cdot75$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·171321 | ·149499 | ·129205 | ·110594 | ·093759 | ·078729 | ·065481 | ·053949 | ·044031 | 0·0 |
| 0·1 | ·167488 | ·146595 | ·127047 | ·109022 | ·092635 | ·077942 | ·064941 | ·053586 | ·043792 | 0·1 |
| 0·2 | ·162875 | ·143049 | ·124373 | ·107044 | ·091201 | ·076922 | ·064231 | ·053101 | ·043467 | 0·2 |
| 0·3 | ·157434 | ·138808 | ·121129 | ·104610 | ·089410 | ·075630 | ·063316 | ·052467 | ·043037 | 0·3 |
| 0·4 | ·151146 | ·133838 | ·117273 | ·101674 | ·087218 | ·074024 | ·062164 | ·051656 | ·042478 | 0·4 |
| 0·5 | ·144026 | ·128133 | ·112784 | ·098208 | ·084592 | ·072073 | ·060742 | ·050640 | ·041766 | 0·5 |
| 0·6 | ·136123 | ·121715 | ·107665 | ·094200 | ·081511 | ·069750 | ·059024 | ·049395 | ·040881 | 0·6 |
| 0·7 | ·127523 | ·114639 | ·101944 | ·089657 | ·077970 | ·067041 | ·056992 | ·047899 | ·039802 | 0·7 |
| 0·8 | ·118347 | ·106989 | ·095677 | ·084613 | ·073983 | ·063949 | ·054637 | ·046141 | ·038514 | 0·8 |
| 0·9 | ·108743 | ·098883 | ·088949 | ·079125 | ·069586 | ·060489 | ·051966 | ·044117 | ·037010 | 0·9 |
| 1·0 | ·098883 | ·090457 | ·081866 | ·073272 | ·064833 | ·056698 | ·048997 | ·041836 | ·035290 | 1·0 |
| 1·1 | ·088949 | ·081866 | ·074555 | ·067151 | ·059797 | ·052627 | ·045765 | ·039316 | ·033363 | 1·1 |
| 1·2 | ·079125 | ·073272 | ·067151 | ·060877 | ·054567 | ·048342 | ·042317 | ·036591 | ·031249 | 1·2 |
| 1·3 | ·069586 | ·064833 | ·059797 | ·054567 | ·049241 | ·043922 | ·038711 | ·033702 | ·028977 | 1·3 |
| 1·4 | ·060489 | ·056698 | ·052627 | ·048342 | ·043922 | ·039451 | ·035017 | ·030702 | ·026584 | 1·4 |
| 1·5 | ·051966 | ·048997 | ·045765 | ·042317 | ·038711 | ·035017 | ·031305 | ·027648 | ·024115 | 1·5 |
| 1·6 | ·044117 | ·041836 | ·039316 | ·036591 | ·033702 | ·030702 | ·027648 | ·024601 | ·021618 | 1·6 |
| 1·7 | ·037010 | ·035290 | ·033363 | ·031249 | ·028977 | ·026584 | ·024115 | ·021618 | ·019143 | 1·7 |
| 1·8 | ·030682 | ·029410 | ·027964 | ·026354 | ·024600 | ·022726 | ·020766 | ·018757 | ·016738 | 1·8 |
| 1·9 | ·025136 | ·024213 | ·023150 | ·021948 | ·020619 | ·019179 | ·017651 | ·016063 | ·014446 | 1·9 |
| 2·0 | ·020352 | ·019696 | ·018929 | ·018049 | ·017061 | ·015975 | ·014807 | ·013575 | ·012303 | 2·0 |
| 2·1 | ·016288 | ·015831 | ·015288 | ·014657 | ·013937 | ·013134 | ·012257 | ·011320 | ·010338 | 2·1 |
| 2·2 | ·012886 | ·012575 | ·012198 | ·011754 | ·011240 | ·010658 | ·010013 | ·009313 | ·008569 | 2·2 |
| 2·3 | ·010081 | ·009872 | ·009617 | ·009310 | ·008950 | ·008537 | ·008071 | ·007559 | ·007006 | 2·3 |
| 2·4 | ·007798 | ·007662 | ·007492 | ·007285 | ·007038 | ·006749 | ·006420 | ·006052 | ·005650 | 2·4 |
| 2·5 | ·005967 | ·005879 | ·005768 | ·005631 | ·005465 | ·005269 | ·005040 | ·004782 | ·004494 | 2·5 |
| 2·6 | ·004516 | ·004461 | ·004390 | ·004302 | ·004192 | ·004061 | ·003906 | ·003727 | ·003526 | 2·6 |

| $k$ | $r=\cdot75$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·035601 | ·028517 | ·022632 | ·017796 | ·013865 | ·010702 | ·008186 | ·006203 | ·004658 | 0·0 |
| 0·1 | ·035447 | ·028420 | ·022572 | ·017760 | ·013843 | ·010690 | ·008179 | ·006199 | ·004656 | 0·1 |
| 0·2 | ·035234 | ·028284 | ·022486 | ·017707 | ·013811 | ·010671 | ·008168 | ·006193 | ·004652 | 0·2 |
| 0·3 | ·034948 | ·028097 | ·022367 | ·017633 | ·013766 | ·010644 | ·008152 | ·006184 | ·004647 | 0·3 |
| 0·4 | ·034570 | ·027847 | ·022205 | ·017530 | ·013702 | ·010605 | ·008129 | ·006171 | ·004639 | 0·4 |
| 0·5 | ·034082 | ·027519 | ·021989 | ·017390 | ·013614 | ·010551 | ·008096 | ·006151 | ·004628 | 0·5 |
| 0·6 | ·033465 | ·027098 | ·021708 | ·017206 | ·013495 | ·010476 | ·008050 | ·006124 | ·004612 | 0·6 |
| 0·7 | ·032702 | ·026569 | ·021348 | ·016967 | ·013340 | ·010377 | ·007988 | ·006086 | ·004589 | 0·7 |
| 0·8 | ·031778 | ·025919 | ·020900 | ·016664 | ·013139 | ·010247 | ·007905 | ·006034 | ·004558 | 0·8 |
| 0·9 | ·030682 | ·025136 | ·020352 | ·016288 | ·012886 | ·010081 | ·007798 | ·005967 | ·004516 | 0·9 |
| 1·0 | ·029410 | ·024213 | ·019696 | ·015831 | ·012575 | ·009872 | ·007662 | ·005879 | ·004461 | 1·0 |
| 1·1 | ·027964 | ·023150 | ·018929 | ·015288 | ·012198 | ·009617 | ·007492 | ·005768 | ·004390 | 1·1 |
| 1·2 | ·026354 | ·021948 | ·018049 | ·014657 | ·011754 | ·009310 | ·007285 | ·005631 | ·004302 | 1·2 |
| 1·3 | ·024600 | ·020619 | ·017061 | ·013937 | ·011240 | ·008950 | ·007038 | ·005465 | ·004192 | 1·3 |
| 1·4 | ·022726 | ·019179 | ·015975 | ·013134 | ·010658 | ·008537 | ·006749 | ·005269 | ·004061 | 1·4 |
| 1·5 | ·020766 | ·017651 | ·014807 | ·012257 | ·010013 | ·008071 | ·006420 | ·005040 | ·003906 | 1·5 |
| 1·6 | ·018757 | ·016063 | ·013575 | ·011320 | ·009313 | ·007559 | ·006052 | ·004782 | ·003727 | 1·6 |
| 1·7 | ·016738 | ·014446 | ·012303 | ·010338 | ·008569 | ·007006 | ·005650 | ·004494 | ·003526 | 1·7 |
| 1·8 | ·014751 | ·012832 | ·011016 | ·009330 | ·007794 | ·006422 | ·005218 | ·004181 | ·003304 | 1·8 |
| 1·9 | ·012832 | ·011253 | ·009740 | ·008317 | ·007005 | ·005818 | ·004765 | ·003847 | ·003063 | 1·9 |
| 2·0 | ·011016 | ·009740 | ·008500 | ·007319 | ·006216 | ·005206 | ·004299 | ·003500 | ·002809 | 2·0 |
| 2·1 | ·009330 | ·008317 | ·007319 | ·006356 | ·005445 | ·004599 | ·003831 | ·003145 | ·002545 | 2·1 |
| 2·2 | ·007794 | ·007005 | ·006216 | ·005445 | ·004705 | ·004009 | ·003369 | ·002790 | ·002277 | 2·2 |
| 2·3 | ·006422 | ·005818 | ·005206 | ·004599 | ·004009 | ·003447 | ·002922 | ·002442 | ·002011 | 2·3 |
| 2·4 | ·005218 | ·004765 | ·004299 | ·003831 | ·003369 | ·002922 | ·002500 | ·002108 | ·001752 | 2·4 |
| 2·5 | ·004181 | ·003847 | ·003500 | ·003145 | ·002790 | ·002442 | ·002108 | ·001795 | ·001506 | 2·5 |
| 2·6 | ·003304 | ·003063 | ·002809 | ·002545 | ·002277 | ·002011 | ·001752 | ·001506 | ·001275 | 2·6 |

| $k$ | $r=\cdot80$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·397583 | ·376613 | ·353778 | ·329412 | ·303929 | ·277783 | ·251453 | ·225411 | ·200091 | $0\cdot0$ |
| $0\cdot1$ | ·376613 | ·358285 | ·338048 | ·316170 | ·293001 | ·268950 | ·244465 | ·220001 | ·195996 | $0\cdot1$ |
| $0\cdot2$ | ·353778 | ·338048 | ·320423 | ·301103 | ·280367 | ·258565 | ·236106 | ·213416 | ·190922 | $0\cdot2$ |
| $0\cdot3$ | ·329412 | ·316170 | ·301103 | ·284337 | ·266086 | ·246639 | ·226344 | ·205592 | ·184786 | $0\cdot3$ |
| $0\cdot4$ | ·303929 | ·293001 | ·280367 | ·266086 | ·250308 | ·233252 | ·215207 | ·196514 | ·177541 | $0\cdot4$ |
| $0\cdot5$ | ·277783 | ·268950 | ·258565 | ·246639 | ·233252 | ·218560 | ·202791 | ·186227 | ·169191 | $0\cdot5$ |
| $0\cdot6$ | ·251453 | ·244465 | ·236106 | ·226344 | ·215207 | ·202791 | ·189259 | ·174834 | ·159790 | $0\cdot6$ |
| $0\cdot7$ | ·225411 | ·220001 | ·213416 | ·205592 | ·196514 | ·186227 | ·174834 | ·162503 | ·149450 | $0\cdot7$ |
| $0\cdot8$ | ·200091 | ·195996 | ·190922 | ·184786 | ·177541 | ·169191 | ·159790 | ·149450 | ·138334 | $0\cdot8$ |
| $0\cdot9$ | ·175878 | ·172850 | ·169028 | ·164320 | ·158664 | ·152028 | ·144430 | ·135932 | ·126647 | $0\cdot9$ |
| $1\cdot0$ | ·153091 | ·150903 | ·148089 | ·144559 | ·140239 | ·135080 | ·129069 | ·122228 | ·114631 | $1\cdot0$ |
| $1\cdot1$ | ·131965 | ·130422 | ·128399 | ·125812 | ·122588 | ·118666 | ·114014 | ·108626 | ·102536 | $1\cdot1$ |
| $1\cdot2$ | ·112660 | ·111601 | ·110181 | ·108330 | ·105979 | ·103065 | ·099544 | ·095393 | ·090617 | $1\cdot2$ |
| $1\cdot3$ | ·095267 | ·094559 | ·093585 | ·092292 | ·090617 | ·088503 | ·085899 | ·082772 | ·079108 | $1\cdot3$ |
| $1\cdot4$ | ·079809 | ·079341 | ·078690 | ·077808 | ·076644 | ·075145 | ·073264 | ·070962 | ·068215 | $1\cdot4$ |
| $1\cdot5$ | ·066233 | ·065934 | ·065509 | ·064922 | ·064132 | ·063095 | ·061768 | ·060113 | ·058101 | $1\cdot5$ |
| $1\cdot6$ | ·054460 | ·054273 | ·054003 | ·053622 | ·053099 | ·052399 | ·051485 | ·050323 | ·048883 | $1\cdot6$ |
| $1\cdot7$ | ·044369 | ·044256 | ·044088 | ·043847 | ·043509 | ·043048 | ·042434 | ·041638 | ·040632 | $1\cdot7$ |
| $1\cdot8$ | ·035820 | ·035753 | ·035651 | ·035503 | ·035290 | ·034993 | ·034591 | ·034058 | ·033373 | $1\cdot8$ |
| $1\cdot9$ | ·028656 | ·028617 | ·028557 | ·028468 | ·028337 | ·028151 | ·027894 | ·027547 | ·027090 | $1\cdot9$ |
| $2\cdot0$ | ·022718 | ·022696 | ·022661 | ·022609 | ·022531 | ·022417 | ·022257 | ·022036 | ·021740 | $2\cdot0$ |
| $2\cdot1$ | ·017847 | ·017835 | ·017816 | ·017786 | ·017741 | ·017673 | ·017575 | ·017438 | ·017251 | $2\cdot1$ |
| $2\cdot2$ | ·013895 | ·013888 | ·013878 | ·013861 | ·013835 | ·013796 | ·013738 | ·013655 | ·013540 | $2\cdot2$ |
| $2\cdot3$ | ·010720 | ·010716 | ·010711 | ·010702 | ·010687 | ·010665 | ·010632 | ·010583 | ·010513 | $2\cdot3$ |
| $2\cdot4$ | ·008195 | ·008194 | ·008191 | ·008186 | ·008178 | ·008166 | ·008147 | ·008119 | ·008078 | $2\cdot4$ |
| $2\cdot5$ | ·006209 | ·006208 | ·006206 | ·006204 | ·006200 | ·006193 | ·006183 | ·006167 | ·006144 | $2\cdot5$ |
| $2\cdot6$ | ·004661 | ·004660 | ·004660 | ·004658 | ·004656 | ·004653 | ·004647 | ·004639 | ·004626 | $2\cdot6$ |

| $k$ | $r=\cdot80$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·175878 | ·153091 | ·131965 | ·112660 | ·095267 | ·079809 | ·066233 | ·054460 | ·044369 | $0\cdot0$ |
| $0\cdot1$ | ·172850 | ·150903 | ·130422 | ·111601 | ·094559 | ·079341 | ·065934 | ·054273 | ·044256 | $0\cdot1$ |
| $0\cdot2$ | ·169028 | ·148089 | ·128399 | ·110181 | ·093585 | ·078690 | ·065509 | ·054003 | ·044088 | $0\cdot2$ |
| $0\cdot3$ | ·164320 | ·144559 | ·125812 | ·108330 | ·092292 | ·077808 | ·064922 | ·053622 | ·043847 | $0\cdot3$ |
| $0\cdot4$ | ·158664 | ·140239 | ·122588 | ·105979 | ·090617 | ·076644 | ·064132 | ·053099 | ·043509 | $0\cdot4$ |
| $0\cdot5$ | ·152028 | ·135080 | ·118666 | ·103065 | ·088503 | ·075145 | ·063095 | ·052399 | ·043048 | $0\cdot5$ |
| $0\cdot6$ | ·144430 | ·129069 | ·114014 | ·099544 | ·085899 | ·073264 | ·061768 | ·051485 | ·042434 | $0\cdot6$ |
| $0\cdot7$ | ·135932 | ·122228 | ·108626 | ·095393 | ·082772 | ·070962 | ·060113 | ·050323 | ·041638 | $0\cdot7$ |
| $0\cdot8$ | ·126647 | ·114631 | ·102536 | ·090617 | ·079108 | ·068215 | ·058101 | ·048883 | ·040632 | $0\cdot8$ |
| $0\cdot9$ | ·116733 | ·106383 | ·095813 | ·085253 | ·074919 | ·065017 | ·055714 | ·047143 | ·039393 | $0\cdot9$ |
| $1\cdot0$ | ·106383 | ·097637 | ·088568 | ·079372 | ·070247 | ·061385 | ·052953 | ·045091 | ·037904 | $1\cdot0$ |
| $1\cdot1$ | ·095813 | ·088568 | ·080933 | ·073072 | ·065156 | ·057357 | ·049835 | ·042733 | ·036160 | $1\cdot1$ |
| $1\cdot2$ | ·085253 | ·079372 | ·073072 | ·066480 | ·059739 | ·052997 | ·046402 | ·040087 | ·034167 | $1\cdot2$ |
| $1\cdot3$ | ·074919 | ·070247 | ·065156 | ·059739 | ·054109 | ·048390 | ·042709 | ·037191 | ·031946 | $1\cdot3$ |
| $1\cdot4$ | ·065017 | ·061385 | ·057357 | ·052997 | ·048390 | ·043632 | ·038832 | ·034098 | ·029531 | $1\cdot4$ |
| $1\cdot5$ | ·055714 | ·052953 | ·049835 | ·046402 | ·042709 | ·038832 | ·034856 | ·030870 | ·026966 | $1\cdot5$ |
| $1\cdot6$ | ·047143 | ·045091 | ·042733 | ·040087 | ·037191 | ·034098 | ·030870 | ·027582 | ·024309 | $1\cdot6$ |
| $1\cdot7$ | ·039393 | ·037904 | ·036160 | ·034167 | ·031946 | ·029531 | ·026966 | ·024309 | ·021620 | $1\cdot7$ |
| $1\cdot8$ | ·032511 | ·031456 | ·030195 | ·028729 | ·027064 | ·025220 | ·023227 | ·021125 | ·018960 | $1\cdot8$ |
| $1\cdot9$ | ·026506 | ·025776 | ·024887 | ·023833 | ·022613 | ·021237 | ·019721 | ·018095 | ·016390 | $1\cdot9$ |
| $2\cdot0$ | ·021353 | ·020860 | ·020248 | ·019508 | ·018635 | ·017631 | ·016506 | ·015275 | ·013962 | $2\cdot0$ |
| $2\cdot1$ | ·017001 | ·016676 | ·016265 | ·015758 | ·015148 | ·014434 | ·013617 | ·012707 | ·011718 | $2\cdot1$ |
| $2\cdot2$ | ·013382 | ·013174 | ·012904 | ·012565 | ·012149 | ·011653 | ·011074 | ·010416 | ·009689 | $2\cdot2$ |
| $2\cdot3$ | ·010417 | ·010286 | ·010114 | ·009893 | ·009616 | ·009279 | ·008879 | ·008415 | ·007892 | $2\cdot3$ |
| $2\cdot4$ | ·008021 | ·007941 | ·007833 | ·007693 | ·007513 | ·007290 | ·007020 | ·006701 | ·006333 | $2\cdot4$ |
| $2\cdot5$ | ·006110 | ·006063 | ·005997 | ·005910 | ·005797 | ·005652 | ·005474 | ·005260 | ·005008 | $2\cdot5$ |
| $2\cdot6$ | ·004607 | ·004579 | ·004540 | ·004488 | ·004417 | ·004327 | ·004212 | ·004072 | ·003904 | $2\cdot6$ |

| $k$ | $r = \cdot 80$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 1 \cdot 8$ | $h = 1 \cdot 9$ | $h = 2 \cdot 0$ | $h = 2 \cdot 1$ | $h = 2 \cdot 2$ | $h = 2 \cdot 3$ | $h = 2 \cdot 4$ | $h = 2 \cdot 5$ | $h = 2 \cdot 6$ | |
| 0·0 | ·035820 | ·028656 | ·022718 | ·017847 | ·013895 | ·010720 | ·008195 | ·006209 | ·004661 | 0·0 |
| 0·1 | ·035753 | ·028617 | ·022696 | ·017835 | ·013888 | ·010716 | ·008194 | ·006208 | ·004660 | 0·1 |
| 0·2 | ·035651 | ·028557 | ·022661 | ·017816 | ·013878 | ·010711 | ·008191 | ·006206 | ·004660 | 0·2 |
| 0·3 | ·035503 | ·028468 | ·022609 | ·017786 | ·013861 | ·010702 | ·008186 | ·006204 | ·004658 | 0·3 |
| 0·4 | ·035290 | ·028337 | ·022531 | ·017741 | ·013835 | ·010687 | ·008178 | ·006200 | ·004656 | 0·4 |
| 0·5 | ·034993 | ·028151 | ·022417 | ·017673 | ·013796 | ·010665 | ·008166 | ·006193 | ·004653 | 0·5 |
| 0·6 | ·034591 | ·027894 | ·022257 | ·017575 | ·013738 | ·010632 | ·008147 | ·006183 | ·004647 | 0·6 |
| 0·7 | ·034058 | ·027547 | ·022036 | ·017438 | ·013655 | ·010583 | ·008119 | ·006167 | ·004639 | 0·7 |
| 0·8 | ·033373 | ·027090 | ·021740 | ·017251 | ·013540 | ·010513 | ·008078 | ·006144 | ·004626 | 0·8 |
| 0·9 | ·032511 | ·026506 | ·021353 | ·017001 | ·013382 | ·010417 | ·008021 | ·006110 | ·004607 | 0·9 |
| 1·0 | ·031456 | ·025776 | ·020860 | ·016676 | ·013174 | ·010286 | ·007941 | ·006063 | ·004579 | 1·0 |
| 1·1 | ·030195 | ·024887 | ·020248 | ·016265 | ·012904 | ·010114 | ·007833 | ·005997 | ·004540 | 1·1 |
| 1·2 | ·028729 | ·023833 | ·019508 | ·015758 | ·012565 | ·009893 | ·007693 | ·005910 | ·004488 | 1·2 |
| 1·3 | ·027064 | ·022613 | ·018635 | ·015148 | ·012149 | ·009616 | ·007513 | ·005797 | ·004417 | 1·3 |
| 1·4 | ·025220 | ·021237 | ·017631 | ·014434 | ·011653 | ·009279 | ·007290 | ·005652 | ·004327 | 1·4 |
| 1·5 | ·023227 | ·019721 | ·016506 | ·013617 | ·011074 | ·008879 | ·007020 | ·005474 | ·004212 | 1·5 |
| 1·6 | ·021125 | ·018095 | ·015275 | ·012707 | ·010416 | ·008415 | ·006701 | ·005260 | ·004072 | 1·6 |
| 1·7 | ·018960 | ·016390 | ·013962 | ·011718 | ·009689 | ·007892 | ·006333 | ·005008 | ·003904 | 1·7 |
| 1·8 | ·016784 | ·014646 | ·012594 | ·010669 | ·008902 | ·007316 | ·005921 | ·004720 | ·003707 | 1·8 |
| 1·9 | ·014646 | ·012904 | ·011204 | ·009584 | ·008074 | ·006698 | ·005470 | ·004400 | ·003484 | 1·9 |
| 2·0 | ·012594 | ·011204 | ·009825 | ·008488 | ·007222 | ·006050 | ·004990 | ·004051 | ·003237 | 2·0 |
| 2·1 | ·010669 | ·009584 | ·008488 | ·007408 | ·006367 | ·005389 | ·004489 | ·003681 | ·002970 | 2·1 |
| 2·2 | ·008902 | ·008074 | ·007222 | ·006367 | ·005530 | ·004730 | ·003981 | ·003298 | ·002688 | 2·2 |
| 2·3 | ·007316 | ·006698 | ·006050 | ·005389 | ·004730 | ·004088 | ·003478 | ·002912 | ·002399 | 2·3 |
| 2·4 | ·005921 | ·005470 | ·004990 | ·004489 | ·003981 | ·003478 | ·002992 | ·002533 | ·002109 | 2·4 |
| 2·5 | ·004720 | ·004400 | ·004051 | ·003681 | ·003298 | ·002912 | ·002533 | ·002168 | ·001826 | 2·5 |
| 2·6 | ·003707 | ·003484 | ·003237 | ·002970 | ·002688 | ·002399 | ·002109 | ·001826 | ·001555 | 2·6 |

| $k$ | $r = \cdot 85$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h = 0 \cdot 0$ | $h = 0 \cdot 1$ | $h = 0 \cdot 2$ | $h = 0 \cdot 3$ | $h = 0 \cdot 4$ | $h = 0 \cdot 5$ | $h = 0 \cdot 6$ | $h = 0 \cdot 7$ | $h = 0 \cdot 8$ | |
| 0·0 | ·411699 | ·390507 | ·367028 | ·341657 | ·314893 | ·287297 | ·259454 | ·231930 | ·205241 | 0·0 |
| 0·1 | ·390507 | ·372323 | ·351791 | ·329205 | ·304980 | ·279618 | ·253671 | ·227700 | ·202237 | 0·1 |
| 0·2 | ·367028 | ·351791 | ·334234 | ·314547 | ·293048 | ·270158 | ·246373 | ·222227 | ·198251 | 0·2 |
| 0·3 | ·341657 | ·329205 | ·314547 | ·297774 | ·279096 | ·258844 | ·237439 | ·215365 | ·193127 | 0·3 |
| 0·4 | ·314893 | ·304980 | ·293048 | ·279096 | ·263239 | ·245704 | ·226826 | ·207018 | ·186743 | 0·4 |
| 0·5 | ·287297 | ·279618 | ·270158 | ·258844 | ·245704 | ·230869 | ·214580 | ·197166 | ·179027 | 0·5 |
| 0·6 | ·259454 | ·253671 | ·246373 | ·237439 | ·226826 | ·214580 | ·200848 | ·185873 | ·169978 | 0·6 |
| 0·7 | ·231930 | ·227700 | ·222227 | ·215365 | ·207018 | ·197166 | ·185873 | ·173295 | ·159674 | 0·7 |
| 0·8 | ·205241 | ·202237 | ·198251 | ·193127 | ·186743 | ·179027 | ·169978 | ·159674 | ·148275 | 0·8 |
| 0·9 | ·179823 | ·177753 | ·174935 | ·171220 | ·166474 | ·160597 | ·153539 | ·145316 | ·136013 | 0·9 |
| 1·0 | ·156018 | ·154636 | ·152704 | ·150090 | ·146664 | ·142314 | ·136961 | ·130575 | ·123180 | 1·0 |
| 1·1 | ·134073 | ·133179 | ·131894 | ·130110 | ·127710 | ·124585 | ·120640 | ·115817 | ·110097 | 1·1 |
| 1·2 | ·114136 | ·113575 | ·112748 | ·111568 | ·109938 | ·107758 | ·104936 | ·101397 | ·097097 | 1·2 |
| 1·3 | ·096270 | ·095929 | ·095413 | ·094657 | ·093583 | ·092109 | ·090151 | ·087631 | ·084490 | 1·3 |
| 1·4 | ·080464 | ·080264 | ·079952 | ·079483 | ·078798 | ·077832 | ·076514 | ·074772 | ·072546 | 1·4 |
| 1·5 | ·066651 | ·066537 | ·066355 | ·066073 | ·065649 | ·065036 | ·064176 | ·063010 | ·061478 | 1·5 |
| 1·6 | ·054719 | ·054656 | ·054553 | ·054388 | ·054135 | ·053758 | ·053215 | ·052458 | ·051437 | 1·6 |
| 1·7 | ·044525 | ·044492 | ·044435 | ·044343 | ·044196 | ·043972 | ·043639 | ·043163 | ·042503 | 1·7 |
| 1·8 | ·035911 | ·035894 | ·035864 | ·035813 | ·035731 | ·035602 | ·035405 | ·035115 | ·034702 | 1·8 |
| 1·9 | ·028707 | ·028699 | ·028684 | ·028657 | ·028612 | ·028540 | ·028428 | ·028257 | ·028006 | 1·9 |
| 2·0 | ·022746 | ·022742 | ·022734 | ·022720 | ·022697 | ·022659 | ·022596 | ·022499 | ·022352 | 2·0 |
| 2·1 | ·017863 | ·017861 | ·017857 | ·017850 | ·017839 | ·017818 | ·017785 | ·017731 | ·017648 | 2·1 |
| 2·2 | ·013903 | ·013902 | ·013900 | ·013897 | ·013891 | ·013881 | ·013864 | ·013835 | ·013789 | 2·2 |
| 2·3 | ·010724 | ·010723 | ·010722 | ·010721 | ·010718 | ·010713 | ·010705 | ·010690 | ·010666 | 2·3 |
| 2·4 | ·008197 | ·008197 | ·008196 | ·008196 | ·008195 | ·008193 | ·008188 | ·008181 | ·008169 | 2·4 |
| 2·5 | ·006210 | ·006210 | ·006209 | ·006209 | ·006209 | ·006208 | ·006206 | ·006202 | ·006196 | 2·5 |
| 2·6 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004660 | ·004659 | ·004658 | ·004655 | 2·6 |

| $k$ | $r=\cdot85$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| $0\cdot0$ | ·179823 | ·156018 | ·134073 | ·114136 | ·096270 | ·080464 | ·066651 | ·054719 | ·044525 | $0\cdot0$ |
| $0\cdot1$ | ·177753 | ·154636 | ·133179 | ·113575 | ·095929 | ·080264 | ·063537 | ·054656 | ·044492 | $0\cdot1$ |
| $0\cdot2$ | ·174935 | ·152704 | ·131894 | ·112748 | ·095413 | ·079952 | ·066355 | ·054553 | ·044435 | $0\cdot2$ |
| $0\cdot3$ | ·171220 | ·150090 | ·130110 | ·111568 | ·094657 | ·079483 | ·066073 | ·054388 | ·044343 | $0\cdot3$ |
| $0\cdot4$ | ·166474 | ·146664 | ·127710 | ·109938 | ·093583 | ·078798 | ·065649 | ·054135 | ·044196 | $0\cdot4$ |
| $0\cdot5$ | ·160597 | ·142314 | ·124585 | ·107758 | ·092109 | ·077832 | ·065036 | ·053758 | ·043972 | $0\cdot5$ |
| $0\cdot6$ | ·153539 | ·136961 | ·120640 | ·104936 | ·090151 | ·076514 | ·064176 | ·053215 | ·043639 | $0\cdot6$ |
| $0\cdot7$ | ·145316 | ·130575 | ·115817 | ·101397 | ·087631 | ·074772 | ·063010 | ·052458 | ·043163 | $0\cdot7$ |
| $0\cdot8$ | ·136013 | ·123180 | ·110097 | ·097097 | ·084490 | ·072546 | ·061478 | ·051437 | ·042503 | $0\cdot8$ |
| $0\cdot9$ | ·125791 | ·114870 | ·103518 | ·092030 | ·080696 | ·069787 | ·059532 | ·050104 | ·041619 | $0\cdot9$ |
| $1\cdot0$ | ·114870 | ·105798 | ·096173 | ·086238 | ·076254 | ·066476 | ·057135 | ·048420 | ·040472 | $1\cdot0$ |
| $1\cdot1$ | ·103518 | ·096173 | ·088208 | ·079814 | ·071209 | ·062623 | ·054275 | ·046359 | ·039031 | $1\cdot1$ |
| $1\cdot2$ | ·092030 | ·086238 | ·079814 | ·072894 | ·065650 | ·058276 | ·050969 | ·043917 | ·037278 | $1\cdot2$ |
| $1\cdot3$ | ·080696 | ·076254 | ·071209 | ·065650 | ·059701 | ·053516 | ·047262 | ·041110 | ·035213 | $1\cdot3$ |
| $1\cdot4$ | ·069787 | ·066476 | ·062623 | ·058276 | ·053516 | ·048456 | ·043230 | ·037982 | ·032855 | $1\cdot4$ |
| $1\cdot5$ | ·059532 | ·057135 | ·054275 | ·050969 | ·047262 | ·043230 | ·038971 | ·034602 | ·030244 | $1\cdot5$ |
| $1\cdot6$ | ·050104 | ·048420 | ·046359 | ·043917 | ·041110 | ·037982 | ·034602 | ·031055 | ·027441 | $1\cdot6$ |
| $1\cdot7$ | ·041619 | ·040472 | ·039031 | ·037278 | ·035213 | ·032855 | ·030244 | ·027441 | ·024518 | $1\cdot7$ |
| $1\cdot8$ | ·034133 | ·033376 | ·032398 | ·031178 | ·029703 | ·027976 | ·026016 | ·023860 | ·021559 | $1\cdot8$ |
| $1\cdot9$ | ·027652 | ·027167 | ·026525 | ·025701 | ·024679 | ·023451 | ·022022 | ·020411 | ·018648 | $1\cdot9$ |
| $2\cdot0$ | ·022138 | ·021838 | ·021428 | ·020890 | ·020203 | ·019356 | ·018345 | ·017174 | ·015863 | $2\cdot0$ |
| $2\cdot1$ | ·017523 | ·017343 | ·017090 | ·016749 | ·016302 | ·015735 | ·015041 | ·014216 | ·013269 | $2\cdot1$ |
| $2\cdot2$ | ·013719 | ·013614 | ·013463 | ·013254 | ·012972 | ·012605 | ·012142 | ·011579 | ·010914 | $2\cdot2$ |
| $2\cdot3$ | ·010627 | ·010568 | ·010481 | ·010357 | ·010184 | ·009954 | ·009656 | ·009283 | ·008831 | $2\cdot3$ |
| $2\cdot4$ | ·008148 | ·008116 | ·008068 | ·007996 | ·007894 | ·007754 | ·007568 | ·007328 | ·007030 | $2\cdot4$ |
| $2\cdot5$ | ·006185 | ·006169 | ·006142 | ·006102 | ·006044 | ·005962 | ·005849 | ·005700 | ·005510 | $2\cdot5$ |
| $2\cdot6$ | ·004650 | ·004641 | ·004627 | ·004606 | ·004574 | ·004527 | ·004461 | ·004371 | ·004253 | $2\cdot6$ |

| $k$ | $r=\cdot85$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·035911 | ·028707 | ·022746 | ·017863 | ·013903 | ·010724 | ·008197 | ·006210 | ·004661 | $0\cdot0$ |
| $0\cdot1$ | ·035894 | ·028699 | ·022742 | ·017861 | ·013902 | ·010723 | ·008197 | ·006210 | ·004661 | $0\cdot1$ |
| $0\cdot2$ | ·035864 | ·028684 | ·022734 | ·017857 | ·013900 | ·010722 | ·008196 | ·006209 | ·004661 | $0\cdot2$ |
| $0\cdot3$ | ·035813 | ·028657 | ·022720 | ·017850 | ·013897 | ·010721 | ·008196 | ·006209 | ·004661 | $0\cdot3$ |
| $0\cdot4$ | ·035731 | ·028612 | ·022697 | ·017839 | ·013891 | ·010718 | ·008195 | ·006209 | ·004661 | $0\cdot4$ |
| $0\cdot5$ | ·035602 | ·028540 | ·022659 | ·017818 | ·013881 | ·010713 | ·008193 | ·006208 | ·004660 | $0\cdot5$ |
| $0\cdot6$ | ·035405 | ·028428 | ·022596 | ·017785 | ·013864 | ·010705 | ·008188 | ·006206 | ·004659 | $0\cdot6$ |
| $0\cdot7$ | ·035115 | ·028257 | ·022499 | ·017731 | ·013835 | ·010690 | ·008181 | ·006202 | ·004658 | $0\cdot7$ |
| $0\cdot8$ | ·034702 | ·028006 | ·022352 | ·017648 | ·013789 | ·010666 | ·008169 | ·006196 | ·004655 | $0\cdot8$ |
| $0\cdot9$ | ·034133 | ·027652 | ·022138 | ·017523 | ·013719 | ·010627 | ·008148 | ·006185 | ·004650 | $0\cdot9$ |
| $1\cdot0$ | ·033376 | ·027167 | ·021838 | ·017343 | ·013614 | ·010568 | ·008116 | ·006169 | ·004641 | $1\cdot0$ |
| $1\cdot1$ | ·032398 | ·026525 | ·021428 | ·017090 | ·013463 | ·010481 | ·008068 | ·006142 | ·004627 | $1\cdot1$ |
| $1\cdot2$ | ·031178 | ·025701 | ·020890 | ·016749 | ·013254 | ·010357 | ·007996 | ·006102 | ·004606 | $1\cdot2$ |
| $1\cdot3$ | ·029703 | ·024679 | ·020203 | ·016302 | ·012972 | ·010184 | ·007894 | ·006044 | ·004574 | $1\cdot3$ |
| $1\cdot4$ | ·027976 | ·023451 | ·019356 | ·015735 | ·012605 | ·009954 | ·007754 | ·005962 | ·004527 | $1\cdot4$ |
| $1\cdot5$ | ·026016 | ·022022 | ·018345 | ·015041 | ·012142 | ·009656 | ·007568 | ·005849 | ·004461 | $1\cdot5$ |
| $1\cdot6$ | ·023860 | ·020411 | ·017174 | ·014216 | ·011579 | ·009283 | ·007328 | ·005700 | ·004371 | $1\cdot6$ |
| $1\cdot7$ | ·021559 | ·018648 | ·015863 | ·013269 | ·010914 | ·008831 | ·007030 | ·005510 | ·004253 | $1\cdot7$ |
| $1\cdot8$ | ·019177 | ·016780 | ·014438 | ·012213 | ·010155 | ·008300 | ·006671 | ·005274 | ·004103 | $1\cdot8$ |
| $1\cdot9$ | ·016780 | ·014858 | ·012937 | ·011073 | ·009314 | ·007698 | ·006253 | ·004992 | ·003919 | $1\cdot9$ |
| $2\cdot0$ | ·014438 | ·012937 | ·011403 | ·009880 | ·008412 | ·007035 | ·005780 | ·004665 | ·003700 | $2\cdot0$ |
| $2\cdot1$ | ·012213 | ·011073 | ·009880 | ·008668 | ·007473 | ·006329 | ·005264 | ·004299 | ·003448 | $2\cdot1$ |
| $2\cdot2$ | ·010155 | ·009314 | ·008412 | ·007473 | ·006526 | ·005599 | ·004716 | ·003900 | ·003166 | $2\cdot2$ |
| $2\cdot3$ | ·008300 | ·007698 | ·007035 | ·006329 | ·005599 | ·004867 | ·004154 | ·003480 | ·002862 | $2\cdot3$ |
| $2\cdot4$ | ·006671 | ·006253 | ·005780 | ·005264 | ·004716 | ·004154 | ·003594 | ·003052 | ·002544 | $2\cdot4$ |
| $2\cdot5$ | ·005274 | ·004992 | ·004665 | ·004299 | ·003900 | ·003480 | ·003052 | ·002628 | ·002221 | $2\cdot5$ |
| $2\cdot6$ | ·004103 | ·003919 | ·003700 | ·003448 | ·003166 | ·002862 | ·002544 | ·002221 | ·001903 | $2\cdot6$ |

| $k$ | $r=\cdot90$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| *0·0* | ·428214 | ·406669 | ·382169 | ·355243 | ·326566 | ·296908 | ·267038 | ·237662 | ·209396 | *0·0* |
| *0·1* | ·406669 | ·388753 | ·367782 | ·344108 | ·318280 | ·290985 | ·262978 | ·235002 | ·207729 | *0·1* |
| *0·2* | ·382169 | ·367782 | ·350403 | ·330201 | ·307559 | ·283038 | ·257324 | ·231146 | ·205211 | *0·2* |
| *0·3* | ·355243 | ·344108 | ·330201 | ·313517 | ·294257 | ·272823 | ·249782 | ·225802 | ·201583 | *0·3* |
| *0·4* | ·326566 | ·318280 | ·307559 | ·294257 | ·278406 | ·260232 | ·240150 | ·218720 | ·196585 | *0·4* |
| *0·5* | ·296908 | ·290985 | ·283038 | ·272823 | ·260232 | ·245325 | ·228353 | ·209730 | ·190001 | *0·5* |
| *0·6* | ·267038 | ·262978 | ·257324 | ·249782 | ·240150 | ·223353 | ·214481 | ·198793 | ·181701 | *0·6* |
| *0·7* | ·237662 | ·235002 | ·231146 | ·225802 | ·218720 | ·209730 | ·198793 | ·186016 | ·171665 | *0·7* |
| *0·8* | ·209396 | ·207729 | ·205211 | ·201583 | ·196585 | ·190001 | ·181701 | ·171665 | ·160017 | *0·8* |
| *0·9* | ·182714 | ·181715 | ·180143 | ·177784 | ·174405 | ·169780 | ·163726 | ·156138 | ·147024 | *0·9* |
| *1·0* | ·157949 | ·157378 | ·156440 | ·154974 | ·152788 | ·149675 | ·145439 | ·139928 | ·133064 | *1·0* |
| *1·1* | ·135313 | ·135001 | ·134466 | ·133596 | ·132243 | ·130238 | ·127399 | ·123560 | ·118595 | *1·1* |
| *1·2* | ·114899 | ·114738 | ·114448 | ·113954 | ·113155 | ·111920 | ·110099 | ·107537 | ·104093 | *1·2* |
| *1·3* | ·096720 | ·096642 | ·096492 | ·096225 | ·095774 | ·095047 | ·093931 | ·092296 | ·090008 | *1·3* |
| *1·4* | ·080721 | ·080685 | ·080611 | ·080473 | ·080230 | ·079822 | ·079169 | ·078170 | ·076716 | *1·4* |
| *1·5* | ·066793 | ·066776 | ·066741 | ·066673 | ·066549 | ·066330 | ·065965 | ·065383 | ·064500 | *1·5* |
| *1·6* | ·054792 | ·054786 | ·054771 | ·054739 | ·054678 | ·054567 | ·054372 | ·054048 | ·053536 | *1·6* |
| *1·7* | ·044563 | ·044560 | ·044554 | ·044540 | ·044511 | ·044459 | ·044358 | ·044186 | ·043903 | *1·7* |
| *1·8* | ·035928 | ·035928 | ·035926 | ·035920 | ·035907 | ·035882 | ·035834 | ·035747 | ·035598 | *1·8* |
| *1·9* | ·028716 | ·028716 | ·028715 | ·028712 | ·028707 | ·028696 | ·028674 | ·028632 | ·028557 | *1·9* |
| *2·0* | ·022749 | ·022749 | ·022749 | ·022749 | ·022746 | ·022742 | ·022732 | ·022713 | ·022677 | *2·0* |
| *2·1* | ·017864 | ·017864 | ·017864 | ·017864 | ·017863 | ·017861 | ·017857 | ·017849 | ·017832 | *2·1* |
| *2·2* | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013901 | ·013897 | ·013890 | *2·2* |
| *2·3* | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010723 | ·010722 | ·010719 | *2·3* |
| *2·4* | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008197 | ·008197 | ·008195 | *2·4* |
| *2·5* | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006209 | *2·5* |
| *2·6* | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | *2·6* |

| $k$ | $r=\cdot90$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| *0·0* | ·182714 | ·157949 | ·135313 | ·114899 | ·096720 | ·080721 | ·066793 | ·054792 | ·044563 | *0·0* |
| *0·1* | ·181715 | ·157378 | ·135001 | ·114738 | ·096642 | ·080685 | ·066776 | ·054786 | ·044560 | *0·1* |
| *0·2* | ·180143 | ·156440 | ·134466 | ·114448 | ·096492 | ·080611 | ·066741 | ·054771 | ·044554 | *0·2* |
| *0·3* | ·177784 | ·154974 | ·133596 | ·113954 | ·096225 | ·080473 | ·066673 | ·054739 | ·044540 | *0·3* |
| *0·4* | ·174405 | ·152788 | ·132243 | ·113155 | ·095774 | ·080230 | ·066549 | ·054678 | ·044511 | *0·4* |
| *0·5* | ·169780 | ·149675 | ·130238 | ·111920 | ·095047 | ·079822 | ·066330 | ·054567 | ·044459 | *0·5* |
| *0·6* | ·163726 | ·145439 | ·127399 | ·110099 | ·093931 | ·079169 | ·065965 | ·054372 | ·044358 | *0·6* |
| *0·7* | ·156138 | ·139928 | ·123560 | ·107537 | ·092296 | ·078170 | ·065383 | ·054048 | ·044186 | *0·7* |
| *0·8* | ·147024 | ·133064 | ·118595 | ·104093 | ·090008 | ·076716 | ·064500 | ·053536 | ·043903 | *0·8* |
| *0·9* | ·136515 | ·124872 | ·112450 | ·099668 | ·086951 | ·074694 | ·063220 | ·052762 | ·043456 | *0·9* |
| *1·0* | ·124872 | ·115490 | ·105164 | ·094224 | ·083047 | ·072009 | ·061453 | ·051649 | ·042786 | *1·0* |
| *1·1* | ·112450 | ·105164 | ·096874 | ·087810 | ·078274 | ·068600 | ·059118 | ·050118 | ·041826 | *1·1* |
| *1·2* | ·099668 | ·094224 | ·087810 | ·080559 | ·072686 | ·064459 | ·056173 | ·048110 | ·040515 | *1·2* |
| *1·3* | ·086951 | ·083047 | ·078274 | ·072686 | ·066410 | ·059641 | ·052617 | ·045591 | ·038805 | *1·3* |
| *1·4* | ·074694 | ·072009 | ·068600 | ·064459 | ·059641 | ·054265 | ·048506 | ·042569 | ·036674 | *1·4* |
| *1·5* | ·063220 | ·061453 | ·059118 | ·056173 | ·052617 | ·048506 | ·043948 | ·039098 | ·034135 | *1·5* |
| *1·6* | ·052762 | ·051649 | ·050118 | ·048110 | ·045591 | ·042569 | ·039098 | ·035274 | ·031231 | *1·6* |
| *1·7* | ·043456 | ·042786 | ·041826 | ·040515 | ·038805 | ·036674 | ·034135 | ·031231 | ·028057 | *1·7* |
| *1·8* | ·035351 | ·034966 | ·034391 | ·033573 | ·032462 | ·031021 | ·029236 | ·027121 | ·024722 | *1·8* |
| *1·9* | ·028427 | ·028216 | ·027887 | ·027399 | ·026709 | ·025777 | ·024576 | ·023097 | ·021354 | *1·9* |
| *2·0* | ·022612 | ·022502 | ·022322 | ·022044 | ·021635 | ·021058 | ·020285 | ·019293 | ·018080 | *2·0* |
| *2·1* | ·017801 | ·017746 | ·017653 | ·017502 | ·017270 | ·016929 | ·016453 | ·015818 | ·015008 | *2·1* |
| *2·2* | ·013876 | ·013850 | ·013803 | ·013725 | ·013600 | ·013408 | ·013128 | ·012738 | ·012222 | *2·2* |
| *2·3* | ·010713 | ·010701 | ·010679 | ·010640 | ·010576 | ·010472 | ·010315 | ·010087 | ·009772 | *2·3* |
| *2·4* | ·008193 | ·008188 | ·008178 | ·008160 | ·008128 | ·008075 | ·007991 | ·007864 | ·007680 | *2·4* |
| *2·5* | ·006208 | ·006206 | ·006202 | ·006194 | ·006179 | ·006153 | ·006110 | ·006042 | ·005940 | *2·5* |
| *2·6* | ·004661 | ·004660 | ·004658 | ·004655 | ·004648 | ·004636 | ·004615 | ·004581 | ·004526 | *2·6* |

| $k$ | $r=\cdot90$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| $0\cdot0$ | ·035928 | ·028716 | ·022749 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot0$ |
| $0\cdot1$ | ·035928 | ·028716 | ·022749 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot1$ |
| $0\cdot2$ | ·035926 | ·028715 | ·022749 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot2$ |
| $0\cdot3$ | ·035920 | ·028712 | ·022749 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot3$ |
| $0\cdot4$ | ·035907 | ·028707 | ·022746 | ·017863 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot4$ |
| $0\cdot5$ | ·035882 | ·028696 | ·022742 | ·017861 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | $0\cdot5$ |
| $0\cdot6$ | ·035834 | ·028674 | ·022732 | ·017857 | ·013901 | ·010723 | ·008197 | ·006210 | ·004661 | $0\cdot6$ |
| $0\cdot7$ | ·035747 | ·028632 | ·022713 | ·017849 | ·013897 | ·010722 | ·008197 | ·006210 | ·004661 | $0\cdot7$ |
| $0\cdot8$ | ·035598 | ·028557 | ·022677 | ·017832 | ·013890 | ·010719 | ·008195 | ·006209 | ·004661 | $0\cdot8$ |
| $0\cdot9$ | ·035351 | ·028427 | ·022612 | ·017801 | ·013876 | ·010713 | ·008193 | ·006208 | ·004661 | $0\cdot9$ |
| $1\cdot0$ | ·034966 | ·028216 | ·022502 | ·017746 | ·013850 | ·010701 | ·008188 | ·006206 | ·004660 | $1\cdot0$ |
| $1\cdot1$ | ·034391 | ·027887 | ·022322 | ·017653 | ·013803 | ·010679 | ·008178 | ·006202 | ·004658 | $1\cdot1$ |
| $1\cdot2$ | ·033573 | ·027399 | ·022044 | ·017502 | ·013725 | ·010640 | ·008160 | ·006194 | ·004655 | $1\cdot2$ |
| $1\cdot3$ | ·032462 | ·026709 | ·021635 | ·017270 | ·013600 | ·010576 | ·008128 | ·006179 | ·004648 | $1\cdot3$ |
| $1\cdot4$ | ·031021 | ·025777 | ·021058 | ·016929 | ·013408 | ·010472 | ·008075 | ·006153 | ·004636 | $1\cdot4$ |
| $1\cdot5$ | ·029236 | ·024576 | ·020285 | ·016453 | ·013128 | ·010315 | ·007991 | ·006110 | ·004615 | $1\cdot5$ |
| $1\cdot6$ | ·027121 | ·023097 | ·019293 | ·015818 | ·012738 | ·010087 | ·007864 | ·006042 | ·004581 | $1\cdot6$ |
| $1\cdot7$ | ·024722 | ·021354 | ·018080 | ·015008 | ·012222 | ·009772 | ·007680 | ·005940 | ·004526 | $1\cdot7$ |
| $1\cdot8$ | ·022113 | ·019390 | ·016660 | ·014023 | ·011568 | ·009357 | ·007428 | ·005794 | ·004445 | $1\cdot8$ |
| $1\cdot9$ | ·019390 | ·017269 | ·015069 | ·012878 | ·010777 | ·008834 | ·007097 | ·005594 | ·004330 | $1\cdot9$ |
| $2\cdot0$ | ·016660 | ·015069 | ·013361 | ·011602 | ·009862 | ·008205 | ·006683 | ·005333 | ·004173 | $2\cdot0$ |
| $2\cdot1$ | ·014023 | ·012878 | ·011602 | ·010242 | ·008850 | ·007482 | ·006188 | ·005009 | ·003970 | $2\cdot1$ |
| $2\cdot2$ | ·011568 | ·010777 | ·009862 | ·008850 | ·007778 | ·006688 | ·005623 | ·004624 | ·003719 | $2\cdot2$ |
| $2\cdot3$ | ·009357 | ·008834 | ·008205 | ·007482 | ·006688 | ·005851 | ·005006 | ·004187 | ·003422 | $2\cdot3$ |
| $2\cdot4$ | ·007428 | ·007097 | ·006683 | ·006188 | ·005623 | ·005006 | ·004360 | ·003712 | ·003087 | $2\cdot4$ |
| $2\cdot5$ | ·005794 | ·005594 | ·005333 | ·005009 | ·004624 | ·004187 | ·003712 | ·003218 | ·002726 | $2\cdot5$ |
| $2\cdot6$ | ·004445 | ·004330 | ·004173 | ·003970 | ·003719 | ·003422 | ·003087 | ·002726 | ·002353 | $2\cdot6$ |

| $k$ | $r=\cdot95$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot0$ | $h=0\cdot1$ | $h=0\cdot2$ | $h=0\cdot3$ | $h=0\cdot4$ | $h=0\cdot5$ | $h=0\cdot6$ | $h=0\cdot7$ | $h=0\cdot8$ | |
| $0\cdot0$ | ·449458 | ·427148 | ·400525 | ·370551 | ·338465 | ·305541 | ·272899 | ·241401 | ·211641 | $0\cdot0$ |
| $0\cdot1$ | ·427148 | ·409885 | ·388048 | ·362178 | ·333273 | ·302578 | ·271347 | ·240657 | ·211315 | $0\cdot1$ |
| $0\cdot2$ | ·400525 | ·388048 | ·371208 | ·350051 | ·325166 | ·297571 | ·268502 | ·239173 | ·210607 | $0\cdot2$ |
| $0\cdot3$ | ·370551 | ·362178 | ·350051 | ·333789 | ·313499 | ·289802 | ·263724 | ·236469 | ·209204 | $0\cdot3$ |
| $0\cdot4$ | ·338465 | ·333273 | ·325166 | ·313499 | ·297954 | ·278690 | ·256354 | ·231956 | ·206660 | $0\cdot4$ |
| $0\cdot5$ | ·305541 | ·302578 | ·297571 | ·289802 | ·278690 | ·263982 | ·245878 | ·225035 | ·202439 | $0\cdot5$ |
| $0\cdot6$ | ·272899 | ·271347 | ·268502 | ·263724 | ·256354 | ·245878 | ·232102 | ·215259 | ·196007 | $0\cdot6$ |
| $0\cdot7$ | ·241401 | ·240657 | ·239173 | ·236469 | ·231956 | ·225035 | ·215259 | ·202487 | ·186977 | $0\cdot7$ |
| $0\cdot8$ | ·211641 | ·211315 | ·210607 | ·209204 | ·206660 | ·202439 | ·196007 | ·186977 | ·175254 | $0\cdot8$ |
| $0\cdot9$ | ·183985 | ·183855 | ·183547 | ·182880 | ·181566 | ·179198 | ·175290 | ·169372 | ·161115 | $0\cdot9$ |
| $1\cdot0$ | ·158631 | ·158584 | ·158461 | ·158172 | ·157551 | ·156332 | ·154150 | ·150570 | ·145180 | $1\cdot0$ |
| $1\cdot1$ | ·135659 | ·135643 | ·135599 | ·135484 | ·135216 | ·134643 | ·133525 | ·131534 | ·128287 | $1\cdot1$ |
| $1\cdot2$ | ·115068 | ·115063 | ·115049 | ·115007 | ·114902 | ·114655 | ·114131 | ·113116 | ·111318 | $1\cdot2$ |
| $1\cdot3$ | ·096800 | ·096799 | ·096794 | ·096781 | ·096743 | ·096646 | ·096422 | ·095949 | ·095036 | $1\cdot3$ |
| $1\cdot4$ | ·080757 | ·080756 | ·080755 | ·080751 | ·080739 | ·080704 | ·080617 | ·080415 | ·079992 | $1\cdot4$ |
| $1\cdot5$ | ·066807 | ·066807 | ·066807 | ·066806 | ·066802 | ·066791 | ·066760 | ·066682 | ·066502 | $1\cdot5$ |
| $1\cdot6$ | ·054799 | ·054799 | ·054799 | ·054799 | ·054798 | ·054795 | ·054785 | ·054757 | ·054688 | $1\cdot6$ |
| $1\cdot7$ | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044564 | ·044561 | ·044552 | ·044528 | $1\cdot7$ |
| $1\cdot8$ | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035929 | ·035927 | ·035919 | $1\cdot8$ |
| $1\cdot9$ | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028716 | ·028716 | ·028716 | ·028713 | $1\cdot9$ |
| $2\cdot0$ | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022749 | $2\cdot0$ |
| $2\cdot1$ | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | $2\cdot1$ |
| $2\cdot2$ | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | $2\cdot2$ |
| $2\cdot3$ | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | $2\cdot3$ |
| $2\cdot4$ | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | $2\cdot4$ |
| $2\cdot5$ | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | $2\cdot5$ |
| $2\cdot6$ | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | $2\cdot6$ |

| $k$ | $r=\cdot95$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
| 0·0 | ·183985 | ·158631 | ·135659 | ·115068 | ·096800 | ·080757 | ·066807 | ·054799 | ·044565 | 0·0 |
| 0·1 | ·183855 | ·158584 | ·135643 | ·115063 | ·096799 | ·080756 | ·066807 | ·054799 | ·044565 | 0·1 |
| 0·2 | ·183547 | ·158461 | ·135599 | ·115049 | ·096794 | ·080755 | ·066807 | ·054799 | ·044565 | 0·2 |
| 0·3 | ·182880 | ·158172 | ·135484 | ·115007 | ·096781 | ·080751 | ·066806 | ·054799 | ·044565 | 0·3 |
| 0·4 | ·181566 | ·157551 | ·135216 | ·114902 | ·096743 | ·080739 | ·066802 | ·054798 | ·044565 | 0·4 |
| 0·5 | ·179198 | ·156332 | ·134643 | ·114655 | ·096646 | ·080704 | ·066791 | ·054795 | ·044564 | 0·5 |
| 0·6 | ·175290 | ·154150 | ·133525 | ·114131 | ·096422 | ·080617 | ·066760 | ·054785 | ·044561 | 0·6 |
| 0·7 | ·169372 | ·150570 | ·131534 | ·113116 | ·095949 | ·080415 | ·066682 | ·054757 | ·044552 | 0·7 |
| 0·8 | ·161115 | ·145180 | ·128287 | ·111318 | ·095036 | ·079992 | ·066502 | ·054688 | ·044528 | 0·8 |
| 0·9 | ·150466 | ·137707 | ·123428 | ·108404 | ·093429 | ·079179 | ·066127 | ·054529 | ·044467 | 0·9 |
| 1·0 | ·137707 | ·128130 | ·116733 | ·104068 | ·090839 | ·077758 | ·065411 | ·054200 | ·044329 | 1·0 |
| 1·1 | ·123428 | ·116733 | ·108208 | ·098130 | ·087009 | ·075480 | ·064166 | ·053577 | ·044044 | 1·1 |
| 1·2 | ·108404 | ·104068 | ·098130 | ·090618 | ·081797 | ·072131 | ·062183 | ·052497 | ·043505 | 1·2 |
| 1·3 | ·093429 | ·090839 | ·087009 | ·081797 | ·075245 | ·067601 | ·059285 | ·050787 | ·042579 | 1·3 |
| 1·4 | ·079179 | ·077758 | ·075480 | ·072131 | ·067601 | ·061945 | ·055389 | ·048305 | ·041121 | 1·4 |
| 1·5 | ·066127 | ·065411 | ·064166 | ·062183 | ·059285 | ·055389 | ·050554 | ·044989 | ·039016 | 1·5 |
| 1·6 | ·054529 | ·054200 | ·053577 | ·052497 | ·050787 | ·048305 | ·044989 | ·040898 | ·036221 | 1·6 |
| 1·7 | ·044467 | ·044329 | ·044044 | ·043505 | ·042579 | ·041121 | ·039016 | ·036221 | ·032796 | 1·7 |
| 1·8 | ·035898 | ·035845 | ·035725 | ·035480 | ·035020 | ·034234 | ·033003 | ·031236 | ·028904 | 1·8 |
| 1·9 | ·028707 | ·028688 | ·028643 | ·028541 | ·028332 | ·027944 | ·027282 | ·026253 | ·024785 | 1·9 |
| 2·0 | ·022747 | ·022742 | ·022726 | ·022687 | ·022600 | ·022425 | ·022100 | ·021550 | ·020695 | 2·0 |
| 2·1 | ·017864 | ·017862 | ·017857 | ·017844 | ·017811 | ·017739 | ·017593 | ·017323 | ·016870 | 2·1 |
| 2·2 | ·013903 | ·013903 | ·013901 | ·013897 | ·013886 | ·013859 | ·013799 | ·013678 | ·013457 | 2·2 |
| 2·3 | ·010724 | ·010724 | ·010723 | ·010722 | ·010719 | ·010710 | ·010687 | ·010638 | ·010540 | 2·3 |
| 2·4 | ·008198 | ·008198 | ·008197 | ·008197 | ·008196 | ·008193 | ·008186 | ·008167 | ·008127 | 2·4 |
| 2·5 | ·006210 | ·006210 | ·006210 | ·006210 | ·006209 | ·006208 | ·006206 | ·006200 | ·006185 | 2·5 |
| 2·6 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004660 | ·004658 | ·004653 | 2·6 |

| $k$ | $r=\cdot95$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·0 |
| 0·1 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·1 |
| 0·2 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·2 |
| 0·3 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·3 |
| 0·4 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·4 |
| 0·5 | ·035930 | ·028716 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·5 |
| 0·6 | ·035929 | ·028716 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·6 |
| 0·7 | ·035927 | ·028716 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·7 |
| 0·8 | ·035919 | ·028713 | ·022749 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·8 |
| 0·9 | ·035898 | ·028707 | ·022747 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·9 |
| 1·0 | ·035845 | ·028688 | ·022742 | ·017862 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·0 |
| 1·1 | ·035725 | ·028643 | ·022726 | ·017857 | ·013901 | ·010723 | ·008197 | ·006210 | ·004661 | 1·1 |
| 1·2 | ·035480 | ·028541 | ·022687 | ·017844 | ·013897 | ·010722 | ·008197 | ·006210 | ·004661 | 1·2 |
| 1·3 | ·035020 | ·028332 | ·022600 | ·017811 | ·013886 | ·010719 | ·008196 | ·006209 | ·004661 | 1·3 |
| 1·4 | ·034234 | ·027944 | ·022425 | ·017739 | ·013859 | ·010710 | ·008193 | ·006208 | ·004661 | 1·4 |
| 1·5 | ·033003 | ·027282 | ·022100 | ·017593 | ·013799 | ·010687 | ·008186 | ·006206 | ·004660 | 1·5 |
| 1·6 | ·031236 | ·026253 | ·021550 | ·017323 | ·013678 | ·010638 | ·008167 | ·006200 | ·004658 | 1·6 |
| 1·7 | ·028904 | ·024785 | ·020695 | ·016870 | ·013457 | ·010540 | ·008127 | ·006185 | ·004653 | 1·7 |
| 1·8 | ·026064 | ·022860 | ·019491 | ·016173 | ·013088 | ·010360 | ·008048 | ·006153 | ·004642 | 1·8 |
| 1·9 | ·022860 | ·020529 | ·017917 | ·015190 | ·012523 | ·010062 | ·007904 | ·006089 | ·004616 | 1·9 |
| 2·0 | ·019491 | ·017917 | ·016024 | ·013917 | ·011731 | ·009609 | ·007666 | ·005975 | ·004566 | 2·0 |
| 2·1 | ·016173 | ·015190 | ·013917 | ·012395 | ·010711 | ·008977 | ·007306 | ·005787 | ·004475 | 2·1 |
| 2·2 | ·013088 | ·012523 | ·011731 | ·010711 | ·009500 | ·008169 | ·006807 | ·005504 | ·004328 | 2·2 |
| 2·3 | ·010360 | ·010062 | ·009609 | ·008977 | ·008169 | ·007214 | ·006172 | ·005114 | ·004108 | 2·3 |
| 2·4 | ·008048 | ·007904 | ·007666 | ·007306 | ·006807 | ·006172 | ·005428 | ·004621 | ·003806 | 2·4 |
| 2·5 | ·006153 | ·006089 | ·005975 | ·005787 | ·005504 | ·005114 | ·004621 | ·004047 | ·003427 | 2·5 |
| 2·6 | ·004642 | ·004616 | ·004566 | ·004475 | ·004328 | ·004108 | ·003806 | ·003427 | ·002989 | 2·6 |

| $k$ | $r=1\cdot0$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
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| $0\cdot1$ | ·460172 | ·460172 | ·420740 | ·382089 | ·344578 | ·308538 | ·274253 | ·241964 | ·211855 | $0\cdot1$ |
| $0\cdot2$ | ·420740 | ·420740 | ·420740 | ·382089 | ·344578 | ·308538 | ·274253 | ·241964 | ·211855 | $0\cdot2$ |
| $0\cdot3$ | ·382089 | ·382089 | ·382089 | ·382089 | ·344578 | ·308538 | ·274253 | ·241964 | ·211855 | $0\cdot3$ |
| $0\cdot4$ | ·344578 | ·344578 | ·344578 | ·344578 | ·344578 | ·308538 | ·274253 | ·241964 | ·211855 | $0\cdot4$ |
| $0\cdot5$ | ·308538 | ·308538 | ·308538 | ·308538 | ·308538 | ·308538 | ·274253 | ·241964 | ·211855 | $0\cdot5$ |
| $0\cdot6$ | ·274253 | ·274253 | ·274253 | ·274253 | ·274253 | ·274253 | ·274253 | ·241964 | ·211855 | $0\cdot6$ |
| $0\cdot7$ | ·241964 | ·241964 | ·241964 | ·241964 | ·241964 | ·241964 | ·241964 | ·241964 | ·211855 | $0\cdot7$ |
| $0\cdot8$ | ·211855 | ·211855 | ·211855 | ·211855 | ·211855 | ·211855 | ·211855 | ·211855 | ·211855 | $0\cdot8$ |
| $0\cdot9$ | ·184060 | ·184060 | ·184060 | ·184060 | ·184060 | ·184060 | ·184060 | ·184060 | ·184060 | $0\cdot9$ |
| $1\cdot0$ | ·158655 | ·158655 | ·158655 | ·158655 | ·158655 | ·158655 | ·158655 | ·158655 | ·158655 | $1\cdot0$ |
| $1\cdot1$ | ·135666 | ·135666 | ·135666 | ·135666 | ·135666 | ·135666 | ·135666 | ·135666 | ·135666 | $1\cdot1$ |
| $1\cdot2$ | ·115070 | ·115070 | ·115070 | ·115070 | ·115070 | ·115070 | ·115070 | ·115070 | ·115070 | $1\cdot2$ |
| $1\cdot3$ | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | $1\cdot3$ |
| $1\cdot4$ | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | $1\cdot4$ |
| $1\cdot5$ | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | $1\cdot5$ |
| $1\cdot6$ | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | $1\cdot6$ |
| $1\cdot7$ | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | $1\cdot7$ |
| $1\cdot8$ | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | $1\cdot8$ |
| $1\cdot9$ | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | $1\cdot9$ |
| $2\cdot0$ | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | $2\cdot0$ |
| $2\cdot1$ | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | $2\cdot1$ |
| $2\cdot2$ | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | $2\cdot2$ |
| $2\cdot3$ | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | $2\cdot3$ |
| $2\cdot4$ | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | $2\cdot4$ |
| $2\cdot5$ | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | $2\cdot5$ |
| $2\cdot6$ | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | $2\cdot6$ |

| $k$ | $r=1\cdot0$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=0\cdot9$ | $h=1\cdot0$ | $h=1\cdot1$ | $h=1\cdot2$ | $h=1\cdot3$ | $h=1\cdot4$ | $h=1\cdot5$ | $h=1\cdot6$ | $h=1\cdot7$ | |
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| $0\cdot1$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot1$ |
| $0\cdot2$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot2$ |
| $0\cdot3$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot3$ |
| $0\cdot4$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot4$ |
| $0\cdot5$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot5$ |
| $0\cdot6$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot6$ |
| $0\cdot7$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot7$ |
| $0\cdot8$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot8$ |
| $0\cdot9$ | ·184060 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $0\cdot9$ |
| $1\cdot0$ | ·158655 | ·158655 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $1\cdot0$ |
| $1\cdot1$ | ·135666 | ·135666 | ·135666 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $1\cdot1$ |
| $1\cdot2$ | ·115070 | ·115070 | ·115070 | ·115070 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $1\cdot2$ |
| $1\cdot3$ | ·096801 | ·096801 | ·096801 | ·096801 | ·096801 | ·080757 | ·066807 | ·054799 | ·044565 | $1\cdot3$ |
| $1\cdot4$ | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·080757 | ·066807 | ·054799 | ·044565 | $1\cdot4$ |
| $1\cdot5$ | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·066807 | ·054799 | ·044565 | $1\cdot5$ |
| $1\cdot6$ | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·054799 | ·044565 | $1\cdot6$ |
| $1\cdot7$ | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | ·044565 | $1\cdot7$ |
| $1\cdot8$ | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | ·035930 | $1\cdot8$ |
| $1\cdot9$ | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | ·028717 | $1\cdot9$ |
| $2\cdot0$ | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | ·022750 | $2\cdot0$ |
| $2\cdot1$ | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | ·017864 | $2\cdot1$ |
| $2\cdot2$ | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | $2\cdot2$ |
| $2\cdot3$ | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | $2\cdot3$ |
| $2\cdot4$ | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | $2\cdot4$ |
| $2\cdot5$ | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | $2\cdot5$ |
| $2\cdot6$ | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | $2\cdot6$ |

| $k$ | $r=1\cdot0$ | | | | | | | | | $k$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | $h=1\cdot8$ | $h=1\cdot9$ | $h=2\cdot0$ | $h=2\cdot1$ | $h=2\cdot2$ | $h=2\cdot3$ | $h=2\cdot4$ | $h=2\cdot5$ | $h=2\cdot6$ | |
| 0·0 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·0 |
| 0·1 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·1 |
| 0·2 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·2 |
| 0·3 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·3 |
| 0·4 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·4 |
| 0·5 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·5 |
| 0·6 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·6 |
| 0·7 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·7 |
| 0·8 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·8 |
| 0·9 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 0·9 |
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| 1·1 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·1 |
| 1·2 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·2 |
| 1·3 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·3 |
| 1·4 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·4 |
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| 1·6 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·6 |
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| 1·8 | ·035930 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·8 |
| 1·9 | ·028717 | ·028717 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 1·9 |
| 2·0 | ·022750 | ·022750 | ·022750 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 2·0 |
| 2·1 | ·017864 | ·017864 | ·017864 | ·017864 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 2·1 |
| 2·2 | ·013903 | ·013903 | ·013903 | ·013903 | ·013903 | ·010724 | ·008198 | ·006210 | ·004661 | 2·2 |
| 2·3 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·010724 | ·008198 | ·006210 | ·004661 | 2·3 |
| 2·4 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·008198 | ·006210 | ·004661 | 2·4 |
| 2·5 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·006210 | ·004661 | 2·5 |
| 2·6 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | ·004661 | 2·6 |

# DOES INSULIN IMPROVE CARBOHYDRATE TOLERANCE IN DIABETES?

## (A STATISTICAL STUDY.)

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It is hardly necessary to dilate upon the value of insulin in the treatment of *Diabetes mellitus*. The physical state and mental attitude of the diabetic, uncontrolled by diet alone, speak for themselves. Death from *uncomplicated* diabetes is and should now be an uncommon occurrence. Diabetics can now be exposed to major surgical procedures of long duration, using anaesthetics of different kinds, and the mortality rate is relatively low. With proper preoperative and postoperative medical care, death due to diabetic coma, and brought on by anaesthesia, is, and should be, a rare occurrence. Insulin has, however, been employed in a sufficiently large number of cases and for a sufficient length of time to enable us to answer a question just as important as, if not of greater importance than, whether the diabetic can be kept alive. Can insulin treatment result in a cure? If not, can it result in a partial cure? And if not, can it at least improve carbohydrate tolerance?

There are no records available, clinical or experimental, which demonstrate a cure. As far as I could ascertain from the literature, there is only one record of an investigation into the question as to whether insulin treatment can result in a partial cure. Harrison (1), at King's College Hospital, London, has been unable to demonstrate a partial remission of the disease in a careful study of five selected cases with observations lasting eleven to eighteen months. All of the five patients needed as much or slightly more insulin at the end of the observation periods. Such fluctuations in dosage of insulin as occurred were attributed to difficulties in balancing the dose of insulin accurately against the diet, variations in the potency of the insulin, and alterations in metabolic activity of the tissues.

We are therefore left with the question as to whether insulin can, at least, *improve* tolerance for carbohydrates. This question is, obviously, just as important as that just discussed. There is no sharp line of demarcation between improvement of carbohydrate tolerance and cure. It is only a matter of degree. If improvement of carbohydrate tolerance could be definitely demonstrated it would be sufficiently encouraging in that partial or real cure may be a function of time. An investigation along these lines was made at the Montreal General Hospital, and it is the purpose of this paper to record the results.

Differing from the unanimity of opinion that insulin does not effect cures, there appear to be differences of opinion as to its ability to improve carbohydrate tolerance. One's experience with diabetes need not be very large before one notices instances which do suggest improvement of tolerance. Diabetics are known to require insulin on admission to the hospital, and before discharge or some time after, are able to reduce the amounts taken or do without it entirely. Also there appears to be anatomical evidence in favour of improvement in tolerance. Boyd and Robinson (2) have reported a case of a child killed in an accident shortly after having been discharged from the hospital, who had improved markedly following insulin treatment. The clinical response to insulin, associated with the anatomical changes found, were strongly suggestive of regeneration of pancreatic tissue. Whether, however, anatomical integrity necessarily parallels functional efficiency is an unsettled problem. We know that such parallelism is not always found in the case of the kidneys.

The greatest difficulties one encounters in interpreting results of an investigation of the kind we have attempted are the possible variations of the disease itself, and the numerous factors which may influence the courses of the different types. One may here repeat an often quoted remark. Sir Rose Bradford put it very well when he stated that "diabetes is not an entity, but a clinical label attached to a number of different conditions with varied origins, different morbid anatomy and liable to follow different courses." The truth of this statement becomes only too evident in the course of the treatment of a series of such patients. For example, there is the acute diabetic. This individual, known to have had no glycosuria, very shortly after some illness, probably of a very mild nature, rapidly develops glycosuria and acidosis and progresses to the stage of coma. With relatively small doses of insulin he is brought out of coma, and in a very short time requires no insulin and takes a fair amount of carbohydrate. In our last case of this type ( 4789—26), this was three weeks. Such patients observed over a period of two or three years do not appear to have made any downward progress. One case I have in mind is now able to take about 250 grams of carbohydrate a day. As is obvious to anyone familiar with food values, this represents a very liberal carbohydrate diet.

There is the "juvenile diabetic," who may run a rather acute course, rapidly develop severe acidosis, and progress to the stage of coma, and, as in the acute diabetes of adults just mentioned, following insulin treatment, may be brought out of coma and eventually require little or no insulin while on fairly liberal diet. One striking example of this type of case was a child of twelve years of age ( 6109—22). Eight days following her recovery from coma, the urine was sugar free, the blood sugar was normal, and it was possible to discontinue the use of insulin.

Then there is the very mild diabetic whose tolerance is suddenly lowered by an infection, and progresses rapidly to the stage of severe acidosis and coma. One such patient developed pneumonia ( 4271—25), and, in spite of taking sixty

units of insulin a day, could barely be kept free of acidosis. Hyperglycaemia and glycosuria were very difficult to control. Following recovery from the pneumonia it was possible to discontinue the use of insulin, and the patient now takes sufficient food to maintain his normal body weight, the urine sugar free and the blood sugar normal.

In view of the very limited knowledge we have of the peculiar metabolism of these three types of the disease just mentioned, and in view of such rapid fluctuations in carbohydrate tolerance as they may show, it is obvious that such patients are not suitable for an investigation of the kind which we have attempted. We are therefore left with the other form of the disease, which, fortunately for this purpose, represents by far the great majority of patients, namely, the chronic (and in the absence of treatment) progressive diabetic. The course of the majority of these patients is very clear. At first the glycosuria which develops is transient, and may be absent for intervals of weeks. As the carbohydrate tolerance is lowered, the glycosuria appears more frequently, and then becomes post-prandial, appearing regularly a few hours after meals. As the disease progresses, the glycosuria becomes persistent, being found throughout the day. With further lowering of carbohydrate tolerance acetone bodies make their appearance, and in time, in the absence of treatment, these patients slowly but surely progress to the stage of coma.

Because of the slow progress and the relative ease with which it is possible to determine the carbohydrate tolerance of these individuals, they make ideal subjects for this study. The problem is, however, not simple. Though we may be dealing with a condition which runs a fairly slow and steady course, and the metabolism of which is fairly understood, there are many variables to consider.

In view of previous observations on the influence of infection, diabetics, though belonging to this group, having foci of infection, cannot be included in such a study. From this group must also be excluded those subjects with a history of gall-bladder disease, i.e. those patients known to have diabetes, definitely pancreatic in origin.

Having selected the subjects, a most difficult factor to exclude is the influence of fasting. The value of starvation in improving carbohydrate tolerance is well known. Though one can avoid fasting, when calculating diets for subjects at rest in bed, by adding to the basal diets the equivalent of about 10 per cent. of the total basal heat production, it is necessary to make additional allowances for the amount, frequency and regularity of exercise when the subjects are out of bed. It is obvious that with an individual not at rest in bed, and on a diet of constant composition, the degree of fasting will depend upon the degree of muscular activity.

Patients whose diets are constructed on the basis of calculated, rather than experimentally determined, caloric requirement may actually be fasting, though the degree of undernutrition may be mild. Continued, however, over a long period

of time such diets will influence the course of the diabetes, in that they will tend to improve carbohydrate tolerance regardless of the insulin administered.

In addition to this there is the difficulty of determining *exactly* how much food patients are receiving. In spite of accurately weighing food we have previously shown (3) that patients may be receiving as much as 25 per cent. either more or less of carbohydrate than is intended unless the diets consist of dairy products only—milk, cream, butter, etc.—of which a large number of analyses have been made and the percentage compositions are fairly accurately known; the compositions of other foods vary widely*.

Then we have the influence of acidosis. Many patients on admission to the hospital have this complication. An increased metabolic rate generally accompanies it. Diets constructed on calculated rather than experimentally determined caloric requirements may therefore fall short of the needs of the individual, *at least temporarily*, and fasting may unintentionally play its part.

Insulin improves the general attitude of the individual, and gives him a brighter outlook on life in general. Muscular tone is increased. This increases metabolism and, providing the diet remains the same, will obviously result in some degree of fasting.

It is obvious from the foregoing observations that such a study as this is rather complicated. It is, at times, possible however, by the application of modern statistical methods, to draw approximately accurate conclusions from what appears to be a very complicated or even at first sight an insolvable problem. Such methods were adopted for this investigation.

Our object was to attempt to demonstrate whether insulin does or does not improve carbohydrate tolerance. Proof of such improvement shall be regarded as having been demonstrated by either one of the following results:

(*a*) The patient, after having been proven to require insulin and having had it for a period of time, can increase the total caloric value and carbohydrate content of the diet and at the same time not increase the dose of insulin, or,

* This fact is recognized by the U.S. Department of Agriculture in publishing the Atwater-Bryant food tables, Bulletin No. 28, but apparently not generally by those employing the same tables in constructing diets. In the introductory remarks to this publication it is stated "In the present publication it is the intention to give the maximum, minimum, and average of all available analyses of American food products up to January 1, 1899, with the exception of milk, butter and other dairy products, and sugars. The number of analyses of such products is so great and the literature of the subject so large that a compilation of the results might appropriately form the subject of a special publication." And later "As a necessary basis of this tabulation, the individual analyses have been collated in detail. In many cases the number of analyses of a single product was considerable, and it is believed that the averages which are given in the tables may be advantageously used in computing the composition of foods used in dietary studies, etc." It will thus be noted that only in the case of dairy products are the average values recommended. This recommendation is obviously based upon one of the theorems of statistics, namely, "the arithmetical mean of a large series of observed values is the most probable value of the quantity measured." Where the number of analyses is small—meats, vegetables, etc.—it is necessary to know more than the average compositions. In spite of these facts, it is generally assumed that the average values given in these tables are the most probable for *all* foods, and they are accepted at face value in calculating diets. [The statistical theorem referred to is, of course, only true for a limited type, the symmetrical, of frequency distributions. K.P.]

(*b*) The patient, after having been proven to require insulin and having had it for a period of time can, without altering the carbohydrate or caloric content of the diet, reduce the amount of insulin taken.

If, when either the dose of insulin has been decreased, and the same diet maintained, or the diet has been increased, on the same dosage of insulin, the urine fails to remain sugar free and the blood sugar normal, this shall be proof that no increase of tolerance has taken place.

Proof of requiring insulin shall be the fact that the individual cannot maintain his body weight, and at the same time keep the blood sugar normal and the urine sugar free on diet alone.

In either case, no other factors, such as fasting, must operate which, by themselves, would tend to improve carbohydrate tolerance.

The first problem with which we were confronted was to eliminate the influence of the diets given insulin patients since *diets resulting in starvation tend to improve carbohydrate tolerance regardless of insulin.* In our clinic, as most likely in all others, the general plans followed in the dietetic management of patients have been those involving either "ladder" or "basal" diets. For insulin patients we at first confined ourselves to basal diets as outlined by the Toronto workers. Since such basal diets, *in the early period of management,* do not allow for energy expended in the usual muscular activities of bed patients, nor for that due to the specific dynamic action of food, it is obvious that though they are not rigid starvation diets, they may result, over a period of time, in some degree of fasting.

The ladder diet which we generally employ is shown in Table I. It will be

TABLE I.

*Ladder Diet.*

| Day | Carbohydrate | Fat | Protein | Calories | Total Calories |
|---|---|---|---|---|---|
| *1* | 15 | 20 | 25 | 340 | 340 |
| *2* | 15 | 20 | 25 | 340 | 680 |
| *3* | 15 | 20 | 25 | 340 | 1020 |
| *4* | 20 | 50 | 30 | 650 | 1670 |
| *5* | 25 | 75 | 35 | 915 | 2585 |
| *6* | 30 | 100 | 40 | 1180 | 3765 |
| *7* | 35 | 125 | 45 | 1445 | 5210 |
| *8* | 40 | 150 | 50 | 1710 | 6920 |
| *9* | 45 | 150 | 50 | 1730 | 8650 |
| *10* | 50 | 150 | 50 | 1750 | 10400 |

noted that after three "green days" the diet is gradually increased to basal caloric requirements. During the period in which the amounts of food are being increased, and the total amounts are still below the basal requirements, it is obvious that the degree of fasting would be much greater than reached with the "basal" type of

diet. With the latter, the degree of fasting is represented only by the deficiency of calories necessary to meet the energy expenditure as a result of muscular activity, and the specific dynamic action of food.

For the purpose of determining quantitatively the difference between the two diets, the records of 500 non-insulin diabetics were investigated. Two hundred and fifty represented "ladder" diets, and two hundred and fifty represented "basal" diets. The average time taken for the patients of each group to become sugar free was calculated. The data necessary for the calculations are shown in Table II. The first column represents the days necessary to render the urines sugar free. The second and third columns represent the incidences of such periods with the

TABLE II.

| Day | Ladder Diet | Basal Diet |
|---|---|---|
| *1* | 16 | 1 |
| *2* | 46 | 4 |
| *3* | 62 | 5 |
| *4* | 34 | 3 |
| *5* | 17 | 8 |
| *6* | 10 | 16 |
| *7* | 12 | 7 |
| *8* | 13 | 18 |
| *9* | 16 | 35 |
| *10* | 4 | 56 |
| *11* | 2 | 29 |
| *12* | 3 | 16 |
| *13* | 5 | 4 |
| *14* | 2 | 11 |
| *15* | 4 | 9 |
| *16* | 2 | 6 |
| *17* | 0 | 12 |
| *18* | 1 | 5 |
| *19* | 1 | 1 |
| *20* | — | 3 |
| *21* | — | 1 |
| Totals | 250 | 250 |
| A.M. | 4·9 | 10·3 |
| S.D. | 3·6 | 3·7 |
| P.E. of Mean | 0·15 | 0·16 |

Difference between means = 5·4

Probable error of difference $= \sqrt{(0{\cdot}15)^2+(0{\cdot}16)^2} = 0{\cdot}22$

$$\frac{\text{Difference}}{\text{Probable error of difference}} = 24{\cdot}5$$

"ladder" and "basal" diets respectively. For each group were then calculated, in order, the standard deviation* and the probable error of the mean. The difference between the means of both groups and the probable error of the difference were then obtained. From the ratio of the difference to the probable error of the difference was determined whether the differences noted between the two diets were due to chance or actually the result of diet. Thus

| | | |
|---|---|---|
| Number of observations of "ladder" diet | 250 | ($N$) |
| Number of observations of "basal" diet | 250 | ($N'$) |
| Average number of days necessary to render urine sugar free with "ladder" diet | 4·9 | ($M$) |
| Average number of days necessary to render urine sugar free with "basal" diet | 10·3 | ($M'$) |
| Standard deviation of "ladder" diet | 3·6 | ($\sigma$) |
| Standard deviation of "basal" diet | 3·7 | ($\sigma_1$) |
| Probable error of $M$ | 0·15 | ($p$) |
| Probable error of $M'$ | 0·16 | ($p'$) |
| Difference between means ($M'-M$) | 5·4 | |
| Probable error of the difference | 0·22 | |

$$\frac{\text{Difference}}{\text{Probable error of difference}} = 24.$$

It is obvious that, in this group of subjects, the individuals, though selected by excluding the three types previously discussed, will still represent the disease in different degrees of severity. This is a complicating variable. They are, however, known not to be very mild, and also known not to require insulin. They belong to what are known in this hospital as Types 3 and 4 previously described (4). That is, they have persistent glycosuria with and without acetonuria. There is no reason, however, on the basis of probabilities, for a larger number representing any particular severity being found in one group more than the other. If a difference should be found between the results of the two diets, statistical treatment of the average values by the above method should demonstrate the part that *chance* may have played in having produced it.

It will be observed that the average period necessary to render the urines sugar free of patients on the "ladder" diet was 4·9 days, whereas on the "basal" diet it was 10·3 days. Now was this difference of five days between these averages caused by the different diets or the result of chance? From the statistical data given it is seen that the difference between these two averages, namely, 5·4, was 24 times the probable error of the difference (i.e. 5·4 ÷ 0·22). In other words it may be stated the chances of the difference being fortuitous or the result of chance were so slight that it is practically certain that the difference is not the result of *random* sampling.

* Standard deviation $= \sqrt{\frac{\Sigma d^2}{N}}$, where $\Sigma d^2$ is the sum of the squares of the residual errors and $N$ is the number of observations.

Probable error of the mean $= 0{\cdot}67449 \frac{\sigma}{\sqrt{N}}$, where $\sigma$ is the standard deviation.

Probable error of the difference $= \sqrt{(P)^2 + (P_1)^2}$, where $P$ and $P_1$ are the probable errors of the means.

Ratio of difference between means to the probable error of the difference $= \frac{M_1 - M}{\sqrt{(P)^2 + (P_1)^2}}$.

Can this difference be attributed to the effects of starvation only? The average basal food requirement of an individual in this group was 1463 calories per 24 hours. If we allow an additional 10 per cent. of this for the muscular activity of bed patients, the daily average requirement is about 1600 calories. It is known that such non-insulin patients become sugar free in an average of three days of *complete starvation.* They therefore become sugar free following a deficit of about 4800 calories. In Table III are shown the daily and total deficits of patients on

TABLE III.

| Day | Food | | | | Calorie Deficit | |
|---|---|---|---|---|---|---|
| | C | F | P | Calories | Daily | Total |
| *1* | 15 | 20 | 25 | 340 | 1415 | 1415 |
| *2* | 15 | 20 | 25 | 340 | 1415 | 2830 |
| *3* | 15 | 20 | 25 | 340 | 1415 | 4245 |
| *4* | 20 | 50 | 30 | 650 | 1105 | 5350 |
| *5* | 25 | 75 | 35 | 915 | 840 | 6190 |
| *6* | 30 | 100 | 40 | 1180 | 575 | 6765 |
| *7* | 35 | 125 | 45 | 1445 | 310 | 7075 |
| *8* | 40 | 150 | 50 | 1710 | 45 | 7120 |
| *9* | 45 | 150 | 50 | 1730 | 25 | 7145 |
| *10* | 50 | 150 | 50 | 1750 | 5 | 7150 |

the ladder diet. In calculating total requirement of individuals taking food, an additional 10 per cent. must be added to the basal requirement for the specific dynamic action of food. The total daily requirement is therefore 1755 calories. It will be observed that a deficit necessary to render the urine sugar free, that is, a deficit of 5265 calories, occurs about the fourth day of treatment. The actual period observed as noted above was about five days.

That caloric deficiency is probably not the only influencing factor is suggested from the data of the basal diets. Assuming an average requirement of 1755 calories per day, the patients on the "basal" diets were fasting to the extent of about 300 calories per day. Since an average deficit of 4800 calories is necessary to render the urine of such patients sugar free, it should have taken about 16 days to accomplish this. The actual time taken was ten days.

The object of the above investigation was to determine, quantitatively, the difference between the effects of diets from the point of view of their ability to render urine sugar free. It will be noted that the "ladder" diet was more effective, but the important fact is that the "basal" diet had also the same effect but required a longer period of time. *The possible influence of such diets improving carbohydrate tolerance must therefore be considered in the interpretation of possible reduction in insulin dosage.*

In our clinic there were 69 cases observed in which after varying periods of time it was possible to reduce the doses of insulin given. A statistical study was

made in order to determine if possible the cause or causes of these reductions. Some of these patients were kept in bed, and others allowed to walk about. Some had acidosis. An attempt was made to determine the influence of these factors. The subjects were therefore divided into the following groups and sub-groups.

Group 1. (*a*) Walking about. (*b*) In bed.

„ 2. (*a*) With acidosis. (*b*) No acidosis.

„ 3. (*a*) No acidosis and walking about. (*b*) No acidosis and in bed.

„ 4. (*a*) Acidosis and walking about. (*b*) Acidosis and in bed.

For each group and sub-group were calculated the average number of days taken before the doses of insulin could be reduced *from the time the diets and doses of insulin were fairly well regulated.* The differences between the average values of the sub-groups of each group were then treated statistically, and the probable error determined. By means of this procedure it was possible somewhat to clarify the picture. The combined data are shown in Table IV.

TABLE IV.

| Group | Number of cases ($N$) | Arithmetical mean days ($M$) noted before insulin could be reduced | Standard deviation ($\sigma$) | $\sqrt{N}$ | $\frac{\sigma}{\sqrt{N}}$ | Probable error of $(M) = 0{\cdot}67449\ \frac{\sigma}{\sqrt{N}}$ | Probable error of difference between means | Difference between means / Probable error of difference |
|---|---|---|---|---|---|---|---|---|
| 1. (*a*) Walking ... ... | 30 | 19·0 | 7·8 | 5·47 | 1·43 | ·95 | 1·77 | 10·9 |
| (*b*) In bed ... ... ... | 39 | 38·4 | 14 | 6·24 | 2·24 | 1·50 | | |
| 2. (*a*) With acidosis ... ... | 40 | 24·1 | 11·1 | 6·32 | 1·75 | 1·17 | 2·30 | 6·0 |
| (*b*) No acidosis ... ... | 29 | 37·9 | 16·0 | 5·38 | 2·96 | 1·98 | | |
| 3 (*a*) No acidosis. Walking | 11 | 23 | 6·5 | 3·31 | 1·96 | 1·31 | 2·39 | 10·1 |
| (*b*) „ „ In bed | 18 | 47 | 12·8 | 4·24 | 3·00 | 2·01 | | |
| 4. (*a*) Acidosis. Walking ... | 19 | 16·6 | 7·2 | 4·35 | 1·65 | 1·10 | 1·86 | 7·7 |
| (*b*) „ In bed ... | 21 | 31·0 | 10·2 | 4·59 | 2·24 | 1·50 | | |

Here again, as with those cases not requiring insulin, the disease was probably represented by various degrees of severity. The argument, however, holds here also that, on the basis of probabilities, there is no reason for a larger number representing any particular severity selecting one group more than another.

It will be noted from the average values in Table IV that in all the groups, "*walking*" and "*acidosis*" appear to be factors influencing the reductions of the amounts of insulin. Since walking and acidosis increase the metabolism of the individual, and since all these subjects were on "basal" diets, they were obviously fasting. The probabilities of the differences in the averages found being the result of chance are shown in the last column of the table. These demonstrate that the differences found were not the result of *random* sampling, but actually due to fasting. For example, the lowest ratio of the difference between the means to the probable error of the difference was 6·0 (Group 2). With this value it can readily be shown that the odds against the occurrence of a deviation as great or greater than 6, as a result of random sampling, are over 1000 to 1 (actually 1341 to 1).

The results of this investigation, as far as it has proceeded, may be briefly summarized.

An attempt was made to determine whether insulin does or does not improve carbohydrate tolerance. In view of the number of variables entering into the problem, the method of approach was statistical.

The general method employed to determine the influence of any particular variable was to divide the subjects into two groups, in one of which the variable studied was excluded as much as possible. The average result in each group was calculated and the difference between the averages noted.

The probable errors of the averages were then calculated and conclusions were drawn from the ratios of the differences between the averages of each group to the probable errors of the differences.

The three variables observed so far were diet, exercise, and acidosis.

In view of the fact that insulin patients were given during the early period of treatment diets calculated on the basis of caloric requirement in the basal metabolic state, the possible influence of such diets on carbohydrate tolerance was investigated. For this purpose the effects of "basal" diets were firstly compared with those of the "ladder" or starvation type in rendering the urines sugar free. It was found that the quantitative effects of "basal" diets continued over a period of time could be measured approximately. The time taken to render urines sugar free on such diets was greatly a function of their deficiencies in caloric values, i.e. a function of the degree of fasting. Since fasting tends to improve carbohydrate tolerance the effects of such basal diets when given in combination with insulin must be considered in the interpretations of possible reductions in the amounts of the latter.

A series of diabetics requiring insulin but who were able in time to reduce the amounts taken were then observed. The subjects were grouped in order to study the influences of some factors, the presence of which would tend to result in fasting and improve carbohydrate tolerance regardless of insulin. The effects of muscular activity and increased metabolism accompanying acidosis, when not

allowed for in the construction of diets, were observed. Though the whole investigation is incomplete, the findings of the effects of the above factors appear to be of sufficient clinical importance to warrant publication. They lead to certain conclusions.

Though one cannot be dogmatic, especially when an inquiry is only in progress, the following interpretation of the above data is suggested:

Insulin does not improve the carbohydrate tolerance of the diabetic of the chronic progressive group. In the majority of cases wherever it was possible to reduce the quantities of insulin taken, other conditions, which are known to lead to improvement of carbohydrate tolerance (fasting, increased muscular activity, etc.), were found.

Though it has been demonstrated statistically that "basal" diets, "exercise" or "increased" metabolism in general could account largely for reductions in the amounts of insulin taken, these do not appear to be the only influencing factors. The individuals in the control groups, in which exercise and acidosis were excluded, were also able to reduce the amounts of insulin taken, but after a longer period of time. Were these reductions due to insulin?

This study emphasizes further, if further emphasis is necessary, the great underlying principle to be followed in the management of the diabetic, namely, undernutrition. It is most tempting, at times, to allow the diabetic more insulin and, thereby, more liberal quantities of food. To do so, in view of our present knowledge, depends upon whether we are seeking immediate results only or those of a more lasting character. *A propos* of this practice I may cite an observation made in a recent editorial (6) by one responsible for insulin: "when depancreatinized dogs treated with insulin are made fat, by feeding them with excess of carbohydrate, they exhibit much more acute symptoms of diabetes when insulin is withdrawn than are observed under the same circumstances in the case of thin dogs. The hyperglycaemia, ketonemia and glycosuria are all more intense, but most striking of all, the general symptoms are extremely acute and a fat animal seldom lives for more than four days after discontinuing the insulin, whereas a thin one may live several weeks."

In our clinic, at the time this paper is being written, there are over 1200 diabetics. The average age of this group is 47 years. At this age the expectation of life is about twenty-four years. That is a long way to go, and our object should be to reach that goal if possible. Until clinical or experimental data are available definitely demonstrating that insulin does improve carbohydrate tolerance, treatment, at least in my opinion, should aim in that direction which tends to do so—namely, construction of diets not only low in carbohydrate but also caloric content, whether the individual does or does not require insulin.

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(6) J. J. R. M. *J. Lab. & Clin. M.* 1927, Vol. XII. p. 719. Editorial on "New Views on Carbohydrate Metabolism."

# A STUDY OF THE AUSTRALIAN AND TASMANIAN SKULLS, BASED ON PREVIOUSLY PUBLISHED MEASUREMENTS.

By G. M. MORANT, D.Sc.

THE most immediate need of biometric craniology is the metrical description of a sufficient number of adequately long series of skulls of different races. Such a description is best furnished by the detailed measurements of a single collection of the same period and from a single restricted locality, but for many racial types no such material is available and another method of proceeding must be used. That is to collect data relating to a larger territory and to pool them together if the evidence warrants the assumption that only one racial type is represented. The present paper deals with the craniology of the natives of Australia and Tasmania in this manner. The only individual measurements considered are those which have been published previously and from them new means and constants of variation have been deduced.

1. *Definitions of Measurements.* Most of the following index letters denoting measurements are ones normally used by workers in the Biometric Laboratory. $C$ = capacity. $L$ = maximum glabella-occipital length in the median sagittal plane. $L'$ = Frankfort projective glabellar horizontal length. $F$ = Flower's ophryo-occipital length. $B$ = maximum calvarial breadth (see p. 436 below). $B'$ = minimum breadth between the temporal lines. $B''$ = greatest breadth of the frontal bone. Bi-stephanic $B$ = chord from stephanion $R$ to stephanion $L$. Bi-asterionic $B$ = chord from asterion $R$ to asterion $L$. $H'$ = basio-bregmatic height. $H$ = Frankfort vertical height from the basion. $OH$ = Frankfort vertical auricular height. $\beta OH$ = auricular height similar to $OH$ but to the bregma. $LB$ = chord nasion to basion. $Q'$ = Frankfort vertical transverse arc from auricular point to auricular point. $\beta Q'$ = transverse arc similar to $Q'$ through the bregma; Broca's transverse arc = arc terminating at the points *sus-auriculaires* and passing through the bregma. $S$ = median sagittal arc from nasion to opisthion. $S_1$ = arc nasion to bregma. $S_2$ = arc bregma to lambda. $S_3$ = arc lambda to opisthion. Arc lambda to inion. Arc inion to opisthion. $S_1'$ = chord nasion to bregma. $S_2'$ = chord bregma to lambda. $S_3'$ = chord lambda to opisthion. Chord lambda to inion. Chord inion to opisthion. $U$ = maximum horizontal circumference passing above the superciliary ridges and through the ophryon. Glabella $U$ = maximum horizontal circumference through the glabella. $PH$ = chord from tip of anterior nasal spine to alveolar point. $G'H$ = chord nasion to alveolar point. $GB$ = chord between the lowest points $R$ and $L$ of the zygomatic-maxillary sutures. $J$ = maximum bizygomatic breadth. External bi-orbital $B$ = maximum breadth between the external orbital processes of the frontal bone at the fronto-malar sutures. Internal

bi-orbital $B$ = chord betweenp oints $R$ and $L$ where the external borders of the orbits meet the fronto-malar sutures. $NH'$ = nasal height from nasion to base of anterior nasal spine. $NH$, $R$ and $L$ = Frankfort nasal height from nasion to lowest edge, $R$ and $L$ of the pyriform aperture. $NB$ = maximum breadth of the pyriform aperture. Height pyriform aperture = from the lowest median point of the nasal bones to the base of the anterior nasal spine. $DC$ = chord from dacryon $R$ to dacryon $L$. Flower's inter-orb. $B$ = chord between the points $R$ and $L$ "where the ridge which forms the posterior boundary of the lacrymal groove meets the fronto-lacrymal suture." $SC$ = minimum chord between naso-maxillary sutures. Flower's $O_1$ = maximum orbital breadth from the point "where the ridge which forms the posterior boundary of the lacrymal groove meets the fronto-lacrymal suture." $O_1'$ = Broca's orbital breadth from the dacryon. $O_1$ = maximum breadth of the orbit. $O_2$ = orbital height whether taken perpendicular to $O_1$ to Flower's $O_1$, or to $O_1'$. (N.B. It was not possible to separate the measurements of the right and left orbits.) Palato-max. $L$ = Flower's maxillary length (*Journal of the Anthropological Institute*, Vol. x. 1880, p. 161) from the alveolar point to "a line drawn across the hinder borders of the maxillary bones." Turner re-named this measurement the "palato-maxillary length" and the title has been generally adopted since. Broca's $G_1$ = palate length from the tip of the posterior nasal spine to a point "derrière les incisives médianes, sur la lèvre postérieure du bord alvéolaire." $G_1'$ = palate length from the base of the posterior nasal spine to an imaginary line tangential to the inner rims of the alveoli of the middle incisors. Palato-max. $B$ = Flower's maxillary width (*loc. cit.*) "between the outer borders of the alveolar arch immediately above the middle of the second molar tooth." Turner re-named this measurement the "palato-maxillary breadth." Broca's $G_2$ = "le maximum d'écartement de la courbe intérieure de l'arcade alvéolaire." $G_2$ = breadth of palate between inner alveolar walls at second molars. $GL$ = chord from basion to alveolar point. *fml* = chord from basion to opisthion. *fmb* = greatest breadth of *foramen magnum*. Nasion-inion chord. Glabella-inion chord. Sub. to gb.-inion chord = the maximum subtense from the glabella-inion chord to the vault in the median sagittal plane.

Various indices were calculated from the above direct measurements. The Occipital Index (Oc. I.), defined to be $100\frac{S_3}{S_3'}\sqrt{\frac{S_3}{24(S_3-S_3')}}$, was found with the aid of Tildesley's table of the function (*Biometrika*, Vol. XIII. 1921, pp. 261—262). Schwalbe's index is the subtense to the glabella-inion chord expressed as a percentage of that chord. $P\angle$ is the Frankfort profile-angle. $N\angle$, $A\angle$ and $B\angle$ are the nasal, alveolar and basal angles of the triangle of which the nasion, alveolar point and basion are the apices. They were found from the chords $G'H$, $GL$ and $LB$ with the aid of Pearson's Trigonometer in the manner described by Fawcett (*Biometrika*, Vol. I. p. 418). The foraminal angle is that between the Frankfort horizontal plane and the chord joining basion to opisthion which forms the sagittal axis of the *foramen magnum*. It is negative if the opisthion is lower than the basion with respect to the horizontal.

2. *Measurements of Australian Skulls.* Published measurements of native Australian skulls were found in the following papers. *A.S.D.* is used as an abbreviation for *Die anthropologischen Sammlungen Deutschlands*, which was published in parts as supplements to the *Archiv für Anthropologie.* Only sexed and adult specimens are referred to.

(*a*) Pruner-Bey: Résultats de craniométrie. *Mémoires de la Société d'Anthropologie de Paris.* T. II. 1865, pp. 417—432. Means are given of 7 ♂ and 3 ♀ skulls in the Muséum d'Histoire Naturelle at Paris, but few of the measurements can be used.

(*b*) J. W. Spengel: *A.S.D.* Göttingen section, 1874. Individual measurements of 19 skulls, pp. 76—79.

(*c*) A. Ecker: *A.S.D.* Freiburg section, 1878. Individual measurements of 4 skulls (Nos. 9—12) on p. 54. The 6 specimens (Nos. 1—6) on pp. 53—54 were omitted in error.

(*d*) C. V. Cauvin: Rapport sur les mensurations et les caractères morphologiques d'une série de crânes australiens. *Archives des Missions Scientifiques et Littéraires. Choix de Rapports et Instructions publié sous les auspices du Ministère de l'Instruction Publique et des Beaux-Arts.* 3ième série. T. VII. 1881, pp. 191—211. Individual indices and horizontal circumferences only of 23 skulls in the Museum at Sydney.

(*e*) W. Turner: Report on the human crania and other bones of the skeletons collected during the voyage of H.M.S. *Challenger,* in the years 1873—1876. *Challenger Reports.* Vol. X. 1884. Individual measurements of 31 skulls in the Anatomical Museum of Edinburgh University are given in Tables III, IV and V.

(*f*) N. de Miklouho-Maclay: Remarks on a skull of an Australian aboriginal from the Lachan District. *Proceedings of the Linnean Society of New South Wales.* Vol. VIII. 1883, pp. 395—396. Another skull of which measurements are given on pp. 401—403 is clearly scaphocephalic.

(*g*) E. Schmidt: *A.S.D.* Leipzig section, 1886. Individual measurements of 5 skulls on pp. 148—149.

(*h*) N. Rüdinger: *A.S.D.* München section, 1892. Individual measurements of 2 skulls on pp. 112—113.

(*i*) C. D. Cooper: Notes on the skull of an aboriginal Australian. *Journal of the Anthropological Institute.* Vol. XXIII. 1893, pp. 153—156.

(*j*) W. L. H. Duckworth: A critical study of the collection of crania of aboriginal Australians in the Cambridge University Museum. *Ibid.* pp. 284—314. Individual measurements of 34 adult skulls in Tables I and II.

(*k*) W. L. H. Duckworth: Notes on skulls from Queensland and South Australia. *Ibid.* Vol. XXIV. 1894, pp. 213—218. Measurements of 3 skulls on pp. 217—218. They are in the Cambridge University Museum.

(*l*) J. T. Wilson: Craniological report upon two skulls collected by the Horn Expedition. Appendix III (pp. 142—148) in: *Report on the work of the Horn*

*scientific Expedition to Central Australia.* Part IV. *Anthropology.* Edited by Baldwin Spencer. London and Melbourne, 1896.

(*m*) W. L. H. Duckworth: Notes on Crania of Australian aborigines. *Journal of the Anthropological Institute.* Vol. XXVII. 1897, pp. 204—208: Measurements of 3 skulls on pp. 206—207. Two were at that time in private collections and the other was in the Cambridge University Museum.

(*n*) V. Giuffrida-Ruggeri: Crani dell' Australia, della Nuova Caledonia e delle isole Salomone. *Atti della Società Romana di Antropologia.* Vol. XII. 1906, pp. 7—35. Measurements of 2 skulls, pp. 13—14.

(*o*) W. H. Flower: *Catalogue of the Specimens illustrating the Osteology and Dentition of vertebrated Animals, recent and extinct, contained in the Museum of the Royal College of Surgeons of England.* Part I, *Man.* 2nd edition, 1907. Individual measurements of 79 adult and sexed skulls are given on pp. 314—337. Measurements are also recorded of a number of specimens for which no sex is given. The first edition of the catalogue (1879) contains part of these data.

(*p*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* 42 Jahrgang, 1910, pp. 176—227. Individual measurements of 121 skulls in the Museum of the Royal College of Surgeons are given on pp. 212—223. They include all the adult and sexed ones of which measurements were provided in Flower's Catalogue of 1907 except Nos. 1070, 1071, 1072, 1074, 1075 and 1093. Basedow determined all Flower's measurements and several additional ones.

(*q*) R. Pöch: Studien an Eingeborenen von Neu-Südwales und an australischen Schädeln. *Mitteilungen der anthropologischen Gesellschaft in Wien.* Band XLV. 1915, pp. 12—94. Individual measurements of 32 adult and sexed skulls belonging to the writer's private collection are given in Tables I—XI. There are 9 adult unsexed and 3 juvenile specimens in the same tables. The measurements are more complete than any other published ones of an Australian series.

(*r*) A. Schultz: Anthropologische Untersuchungen an der Schädelbasis. *Archiv für Anthropologie.* Neue Folge, Band XVI. 1918, pp. 1—103. Individual measurements of 38 skulls are given on pp. 56—63. The specimens are in the following museums: Dresden (Zool.-antr.-ethn. Mus.) 27, Berlin (Anatomie) 7, München (Anthrop. Institut) 2, Zürich (Anthrop. Institut) 2.

3. *A Comparison of Australian Skulls from different Localities.* From the sources given in Section 2 above it was possible to collect measurements of about 300 sexed Australian skulls. In the case of the greater number of them the localities from which they had been obtained were also recorded. The smallest groupings which could be used conveniently were those given by the six territorial divisions of the continent and more than three-quarters of the skulls could be assigned to one or other of those divisions. The most reliable male means that can be given for the groups are in Table I. Before pooling the measurements provided by different workers great care was taken to ensure that they had been

determined by using precisely similar methods of technique and many insufficiently defined ones had to be excluded. Exceptions to that practice were only made in the case of two characters: all capacities were accepted, although the different methods used to determine them might well have led to substantially different results, and all orbital heights were pooled whether they had been taken perpendicular to Flower's, the dacryal or some other orbital breadth. This latter pooling may be considered a fairly safe procedure. In general, the importance of comparing only those measurements which have been determined by identical methods is evident enough, but, as so often in other fields, that precaution has been neglected by many who have contributed to the craniometry of Australian peoples.

The means of each of the divisions of Table I were calculated from the individual measurements provided by several of the writers cited. The numbers of skulls are as small as any that could be used with safety, but, in spite of that

TABLE I.

*Mean Measurements (Males) of Groups of Australian Skulls.*

| Character | Queensland | New South Wales | Victoria | South Australia | Western Australia | Northern Territory |
|---|---|---|---|---|---|---|
| *C* ... | 1287·7 (19) | 1287·7 (38) | 1311·3 (18) | 1319·6 (39) | 1255·5 (20) | 1224·2 (18) |
| *F* ... | 185·1 (17) | 187·0 (22) | 186·9 (16) | 187·0 (43) | 186·9 (15) | 180·2 (17) |
| *B* ... | 131·1 (19) | 132·6 (43) | 132·3 (18) | 132·8 (47) | 130·8 (22) | 128·3 (18) |
| *B'* ... | 95·2 (20) | 96·7 (36) | 96·7 (17) | 96·1 (37) | 96·8 (15) | 94·3 (17) |
| *LB* ... | 101·3 (19) | 102·7 (27) | 103·8 (16) | 102·4 (41) | 100·7 (22) | 103·8 (18) |
| *H'* ... | 133·3 (20) | 136·1 (29) | 134·3 (16) | 131·5 (45) | 132·6 (22) | 132·8 (18) |
| *S* ... | 375·7 (9) | 372·8 (26) | 379·2 (6) | 369·4 (22) | 373·0 (12) | 361·1 (14) |
| *U* ... | 510·2 (22) | 518·1 (35) | 517·5 (14) | 518·2 (43) | 507·3 (11) | 501·9 (16) |
| *fml* ... | 34·4 (18) | 34·7 (27) | 35·3 (15) | 35·8 (36) | 35·1 (22) | 35·6 (18) |
| *fmb* ... | 29·1 (15) | 29·3 (22) | 30·4 (11) | 30·9 (34) | 30·2 (17) | 30·4 (17) |
| *J* ... | 131·9 (19) | 133·4 (36) | 135·8 (15) | 134·6 (35) | 133·2 (22) | 134·2 (17) |
| *NH'* ... | 49·1 (19) | 49·6 (19) | 49·8 (16) | 50·1 (37) | 48·1 (15) | 49·0 (17) |
| *NB* ... | 26·4 (20) | 26·5 (20) | 27·5 (16) | 27·1 (37) | 26·1 (15) | 27·6 (17) |
| $O_2$ ... | 34·0 (20) | 33·6 (23) | 33·6 (16) | 33·4 (32) | 33·3 (15) | 33·4 (17) |
| $100\,B/F$ ... | 70·5 (17) | 70·8 (21) | 70·6 (16) | 71·2 (43) | 70·4 (16) | 71·3 (17) |
| $100\,H'/F$ ... | 72·3 (17) | 73·0 (21) | 72·2 (16) | 70·6 (41) | 70·6 (15) | 74·0 (17) |
| $100\,B/H'$ ... | 98·6 (25) | 97·6 (34) | 98·3 (16) | 101·1 (47) | 98·7 (22) | 96·7 (18) |
| $100\,fmb/fml$ | 84·5 (16) | 84·2 (22) | 85·7 (11) | 86·4 (34) | 85·4 (17) | 85·3 (17) |
| $100\,NB/NH'$ | 54·7 (25) | 54·2 (24) | 55·3 (16) | 53·8 (39) | 54·7 (16) | 56·7 (17) |

fact, the means appear to show extraordinarily small differences. The cephalic indices ($100\,B/F$) are remarkably constant and the most discordant measurement would seem to be the small length ($F$) of the Northern Territory skulls. But little can be said without a knowledge of whether the differences are significant or not and, in any case, the evidence provided by a single measurement can be of little value. Although the material is not ample, a trustworthy comparison may be made by using the biometric method of the coefficient of racial likeness *. By taking account of a number of characters at the same time—measuring the differences in terms of their statistical significance—that criterion

* Karl Pearson: On the Coefficient of Racial Likeness. *Biometrika*, Vol. XVIII. 1926, pp. 105—117.

largely compensates for the small sizes of the samples. The standard deviations of characters are needed for that purpose and, as none have as yet been given for Australian skulls, the values for the 26th—30th Dynasty series provided by Davin and Pearson* may be used provisionally. They are the most reliable constants of variability available and also almost the smallest recorded as yet for a racial series of skulls. By using the Egyptian standard deviations for Australian groups the resulting coefficients are likely to be larger than the true ones, so we shall be erring on the side of caution. The coefficients in Table II were calculated for the 19 characters given in Table I. When it is remembered that the samples were drawn from exceedingly large areas, the lowness of the coefficients is seen to be a most noteworthy fact. There is perfect statistical justification for considering that the three eastern groups (Queensland, New South Wales and Victoria) represent a single homogeneous population. The skulls from Western Australia cannot be distinguished from that population and the South Australian group is but slightly differentiated from it. The latter stands closest to the groups from the adjoining regions of Western Australia and Victoria, but its bond with the New South Wales and Queensland skulls is close enough to suggest that we are only dealing with variants of a single racial type and not with discrete races. The Northern Territory skulls, however, are sufficiently removed from all the others to suggest a racial difference. Even so, that difference is not greater than one which may be found between two neighbouring and closely related races in an ethnically heterogeneous area such as Europe. It may be noticed that the group furthest removed from the Northern Territory one is that from South Australia, so the slight divergence of the latter from the type found in Queensland, New South Wales, Victoria and Western Australia cannot be supposed due to admixture with the differing type found in the north. From a comparison of the $a$'s† for

TABLE II.

*Coefficients of Racial Likeness between Groups of Male Australian Skulls*[1].

| | Queensland (18·7) | New South Wales (27·6) | Victoria (15·1) | South Australia (38·5) | Western Australia (17·4) | Northern Territory (17·1) |
|---|---|---|---|---|---|---|
| Queensland (18·7)[2] ... | — | −0·06 ± ·21 | 0·26 ± ·21 | 1·43 ± ·21 | −0·34 ± ·21 | 1·51 ± ·21 |
| New South Wales (27·6) | −0·06 ± ·21 | — | −0·18 ± ·21 | 1·96 ± ·21 | 0·61 ± ·21 | 2·89 ± ·21 |
| Victoria (15·1) ... ... | 0·26 ± ·21 | −0·18 ± ·21 | — | 0·18 ± 21 | 0·55 ± ·21 | 1·99 ± ·21 |
| South Australia (38·5) ... | 1·43 ± ·21 | 1·96 ± ·21 | 0·18 ± ·21 | — | 0·86 ± ·21 | 4·40 ± ·21 |
| Western Australia (17·4) | −0·34 ± ·21 | 0·61 ± ·21 | 0·55 ± ·21 | 0·86 ± ·21 | — | 1·83 ± ·21 |
| Northern Territory (17·1) | 1·51 ± ·21 | 2·89 ± ·21 | 1·99 ± ·21 | 4·40 ± ·21 | 1·83 ± ·21 | — |

[1] All the coefficients are based on the 19 characters given in Table I.

[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

* Karl Pearson and Adelaide G. Davin: On the Biometric Constants of the Human Skull. *Biometrika*, Vol. xvi. 1924, pp. 328—363.

† With the usual notation $a = \frac{n_s n_{s'}}{n_s + n_{s'}} \left( \frac{M_s - M_{s'}}{\sigma_s} \right)^2$.

individual characters, it is seen that the divergence of the Northern Territory type from all the others is due chiefly to the smaller size of nearly all its calvarial direct measurements and particularly *F*, *B*, *U* and *S*. The Northern Territory means are also smallest for the capacity (*C*), but not for *LB* and *H'*. The aberrant type has extreme values for the indices 100 *H'*/*F* and 100 *B*/*H'* and they show some significant differences, but none whatever are found between facial measurements. The divergence between the South Australian and New South Wales series is due to *H'* and the indices involving *H'* (100 *H'*/*F* and 100 *B*/*H'*): *fmb* is the only other character showing a significant difference.

It might be thought that the evidence of Table II would be sufficient to justify the pooling of all the Australian skulls except those from the Northern Territory. It was felt safer, however, to re-group the material before making that assumption. Two roughly equal groups were made by clubbing the Western Australian, South Australian and Victorian skulls together to give one, and those from Queensland and New South Wales to give another. Both ♂ and ♀ means are in Table III and with them are the measurements of the short series of Australian skulls from unknown localities. The ♀ values for the Northern Territory specimens are in the same table, the ♂ having been given already in Table I. Coefficients for the two sexes between the three groups of skulls from known localities are in Table IV. The southern and eastern populations have the

TABLE III.

*Mean Measurements of Groups of Australian Skulls.*

| Character | Male | | | Female | | | |
|---|---|---|---|---|---|---|---|
| | Western Australia, South Australia and Victoria | Queensland and New South Wales | Unknown localities | Western Australia, South Australia and Victoria | Queensland and New South Wales | Northern Territory | Unknown localities |
| *C* ... ... | 1301·0 (77) | 1287·7 (57) | 1310·9 (30) | 1149·9 (39) | 1141·4 (27) | 1142·5 (10) | 1149·4 (9) |
| *F* ... ... | 187·0 (74) | 186·2 (39) | 183·8 (22) | 176·6 (31) | 175·4 (22) | 171·3 (10) | 173·4 (5) |
| *B* ... ... | 132·2 (87) | 132·1 (62) | 132·3 (39) | 127·3 (43) | 127·1 (30) | 124·5 (10) | 129·3 (12) |
| *B'* ... ... | 96·4 (69) | 96·2 (56) | 96·1 (27) | 91·5 (30) | 91·7 (26) | 91·2 (10) | 91·8 (8) |
| *LB* ... ... | 102·2 (79) | 102·1 (46) | 101·0 (30) | 95·2 (36) | 94·9 (28) | 93·4 (10) | 95·8 (9) |
| *H'* ... ... | 132·3 (83) | 134·9 (49) | 135·1 (29) | 124·9 (40) | 126·5 (29) | 128·1 (10) | 127·7 (6) |
| *S* ... ... | 372·0 (40) | 373·5 (35) | 373·8 (13) | 352·6 (21) | 353·9 (16) | 345·2 (5) | 358·5 (4) |
| *U* ... ... | 516·3 (68) | 515·1 (57) | 510·7 (33) | 489·2 (27) | 488·3 (31) | 477·5 (10) | 492·6 (11) |
| *fml* ... ... | 35·5 (73) | 34·6 (45) | 35·7 (34) | 34·2 (35) | 33·1 (28) | 33·1 (10) | 32·6 (9) |
| *fmb* ... ... | 30·6 (62) | 29·2 (37) | 30·1 (34) | 28·7 (29) | 28·7 (23) | 28·0 (10) | 28·4 (9) |
| *J* ... ... | 134·4 (72) | 132·9 (55) | 134·0 (37) | 122·4 (36) | 122·6 (28) | 118·9 (9) | 121·3 (9) |
| *NH'* ... | 49·6 (68) | 49·4 (38) | 48·5 (28) | 46·1 (27) | 45·0 (21) | 43·6 (10) | 45·0 (8) |
| *NB* ... ... | 27·0 (68) | 26·4 (40) | 27·4 (23) | 25·1 (27) | 25·9 (23) | 24·7 (10) | 27·6 (5) |
| $O_2$ ... ... | 33·4 (63) | 33·8 (43) | 33·4 (13) | 33·1 (26) | 32·7 (24) | 31·4 (10) | 30·7 (8) |
| 100 *B*/*F* ... | 70·9 (75) | 70·7 (38) | 71·9 (22) | 72·6 (31) | 72·1 (22) | 72·8 (10) | 73·8 (5) |
| 100 *H'*/*F* ... | 71·0 (72) | 72·7 (38) | 73·6 (22) | 70·4 (30) | 72·4 (14) | 74·9 (10) | 73·8 (5) |
| 100 *B*/*H'* ... | 100·0 (85) | 98·0 (59) | 98·0 (29) | 102·3 (40) | 99·8 (37) | 97·2 (10) | 100·0 (6) |
| 100 *fmb*/*fml* | 86·0 (62) | 84·3 (38) | 84·3 (34) | 84·1 (29) | 85·7 (24) | 84·7 (10) | 86·9 (8) |
| 100 *NB*/*NH'* | 54·3 (71) | 54·5 (49) | 56·5 (21) | 54·4 (27) | 56·7 (30) | 56·8 (10) | 60·9 (5) |

♂ coefficient of 1·58 ± ·21 and the ♀ of 0·69 ± ·21*. The former differs quite significantly from zero, but it indicates a bond of affinity which is decidedly closer than one normally found between two distinct but neighbouring races†. The ♀ coefficient differs from zero by less than four times its probable error. Between the ♂ means the only significant differences ($\alpha > 6$ say) are for *fmb* ($\alpha = 9{\cdot}82$), 100 $H'/F$ (8·31), $H'$ (8·23) and 100 $B/H'$ (7·52): 100 $B/H'$ ($\alpha = 7{\cdot}25$) is the only ♀ character differentiated. From this evidence, it was supposed legitimate to conclude that the native population of the whole of Australia, with the exception of the Northern Territory, belongs to a single racial type which shows some slight local variation. That will be called, for convenience, the *A* type. The material considered is not sufficient to provide a reliable determination of the differences which may exist between sub-groups of the *A* type. The Northern Territory skulls

TABLE IV.

*Coefficients of Racial Likeness between Groups of Australian Skulls*[1].

| | Male | | | | Female | | | |
|---|---|---|---|---|---|---|---|---|
| | | Western Australia, South Australia and Victoria (70·9)† | Queensland and New South Wales (46·4) | Northern Territory (17·1) | | Western Australia, South Australia and Victoria (31·8) | Queensland and New South Wales (25·4) | Northern Territory (9·7) |
| Western Australia, South Australia and Victoria | (70·9)[2] | — | 1·58 ± ·21 | 3·64 ± ·21 | (31·8)[2] | — | 0·69 ± ·21 | 3·36 ± ·21 |
| Queensland and New South Wales | (46·4) | 1·58 ± ·21 | — | 3·03 ± ·21 | (25·4) | 0·69 ± ·21 | — | 1·22 ± ·21 |
| Northern Territory | (17·1) | 3·64 ± ·21 | 3·03 ± ·21 | — | (9·7) | 3·36 ± ·21 | 1·22 ± ·21 | — |

[1] All the coefficients are based on the 19 characters given in Tables I and III.

[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

* As in the writer's opinion, it is most reasonable to suppose that this ♂ coefficient is greater than the ♀ because the ♂ means are based on larger numbers of skulls, so it would be fallacious to assume that the ♂ types are less closely related than the ♀ ones. He believes that the coefficients ought not to be compared one with another unless allowance is made for the fact that the series concerned are made up by different numbers of skulls. Admitting that necessity, we have a reasonable explanation of the fact that all the ♂ coefficients in Table IV and in Table V below are greater than the corresponding ♀ coefficients. Again it is not disconcerting to find that the coefficients with the Northern Territory means become greater according as the means with which they are compared are based on larger numbers of skulls (Tables II, IV and V). The incidental evidence adduced in the present paper is quite insufficient to decide this question. It will make little difference to the conclusions arrived at here whether one view or the other be taken.

† Between two ♂ 17th-century London series, the Farringdon Street (91·9) and Whitechapel (90·3), Miss Hooke found a coefficient of 4·15 ± ·18 (*Biometrika*, Vol. xviii. 1926, p. 24). Coefficients less than 5 are seldom found between European series of that length.

are sufficiently distinguished from all the others to make it necessary to assign them to a distinct, but closely allied, race. Their significant coefficients (Table IV) are occasioned almost entirely by their small calvarial direct measurements *F*, *B*, *S* and *U*, judging by both ♂ and ♀ means. The indices 100 *H'*/*F* and 100 *B*/*H'* also show some significant differences, but none are found between the basio-bregmatic heights (*H'*), the nasio-basion lengths (*LB*), the cephalic indices or the facial measurements. The pooled means of the skulls of the *A* type for all available characters are given in Table VI and the coefficients computed for the 19 characters hitherto considered (i.e. the ones in Tables I and III) are in Table V. The small groups of Australian skulls from unknown localities are compared with the others in the same table. Between the Northern Territory and *A* type Australians there is a ♂ coefficient of 3·69 ± ·21 and the significant differences are confined to six of the characters compared, viz. *F* (α = 17·56), *U* (14·17), *B* (10·85), 100 *H'*/*F* (9·95), *S* (9·84) and *C* (6·17): the ♀ coefficient is 2·83 ± ·21 and the only α's greater than 6 are for 100 *H'*/*F* (16·18), *F* (8·56), 100 *B*/*H'* (8·56) and *U* (7·84). As it was reasonable to expect, the means of the skulls from unknown localities are intermediate between those of the two distinct racial types.

TABLE V.

*Coefficients of Racial Likeness between Groups of Australian Skulls*[1].

| | Male | | | | Female | | | |
|---|---|---|---|---|---|---|---|---|
| | | Australian *A*[3] (129·6)[2] | Unknown localities (27·4) | Northern Territory (17·1) | | Australian *A*[3] (58·9) | Unknown localities (7·4) | Northern Territory (9·7) |
| Australian *A*[3] ... | (129·6)[2] | — | 0·93 ± ·21 | 3·69 ± ·21 | (58·9)[2] | — | 0·88 ± ·21 | 2·83 ± ·21 |
| Unknown localities | (27·4) | 0·93 ± ·21 | — | 1·27 ± ·21 | (7·4) | 0·88 ± ·21 | — | 0·74 ± ·21 |
| Northern Territory | (17·1) | 3·69 ± ·21 | 1·27 ± ·21 | — | (9·7) | 2·83 ± ·21 | 0·74 ± ·21 | — |

[1] All the coefficients are based on the 19 characters given in Tables I and III.

[2] The numbers in brackets are the mean numbers of skulls available for the characters used in computing the coefficients.

[3] The *A* group is made up by pooling the skulls from Queensland, New South Wales, Victoria, South Australia and Western Australia.

4. *Measurements of Tasmanian skulls.* Published measurements of native Tasmanian adult skulls were found in the following papers:

(*a*) A. de Quatrefages and E. T. Hamy: *Crania Ethnica. Les crânes des races humaines.* Paris, 1882. Measurements of 1 ♀ skull (*Tasmanien du nord*) are given on p. 228. The three series of means of small groups in the same table could not be used as the measurements are in whole mms. and there is no indication of the number of specimens on which each mean is based.

(*b*) W. R. Harper and A. H. Clarke: Notes on the measurements of the Tasmanian Crania in the Tasmanian Museum, Hobart. *Papers and Proceedings*

*of the Royal Society of Tasmania for 1897.* 1898, pp. 97—110. Of the 19 skulls in the Museum described as Tasmanians, the writers of this paper rejected three as being incorrectly classed. After comparing the measurements, they decided to reject three others (Nos. 1 *a*, 2 *a* and 3 *a*), supposing them to be half-castes. No. 1 *a* has a nasal index (100 *NB*/*NH*′) of 42·8 and a gnathic index (100 *GL*/*LB*) of 93·4; No. 2 *a* has a nasal index of 44·2 and the gnathic index could not be found; No. 3 *a* has a nasal index of 53·2 and a gnathic index of 97·0. For 86 Tasmanian skulls of which the origin has not been questioned the lowest nasal index recorded is 49·0, and for 81 skulls the lowest gnathic index is 94·2. The exclusion of Nos. 1 *a* and 2 *a* would seem to be quite justified, but all the measurements of No. 3 *a* are in fair agreement with the Tasmanian means and there seems to be no sufficient reason for rejecting it*. Of the adult and sexed skulls measured by Harper and Clarke, Nos. 1—12 and 3 *a* were accepted.

(*c*) W. L. H. Duckworth: Craniological Notes on the Aborigines of Tasmania. *Journal of the Anthropological Institute.* Vol. XXXII. 1902, pp. 171—181. Individual measurements of three specimens in the Cambridge Anatomical Museum are given.

(*d*) H. Klaatsch: Bericht über einen anthropologischen Streifzug nach London und auf das Plateau von Süd-England. *Zeitschrift für Ethnologie.* Bd. XXXV. 1903, pp. 875—920. A few unusual measurements of 8 skulls in London and Paris museums are given in Tables I—V.

(*e*) Sir William Turner: The Craniology, Racial Affinities and Descent of the Aborigines of Tasmania. *Transactions of the Royal Society of Edinburgh.* Vol. XLVI. 1908, pp. 365—403. Individual measurements of 9 adult skulls in Edinburgh museums are given on p. 367; measurements of six other specimens in the Oxford University Museum are on p. 375†.

(*f*) Sir William Turner: The Aborigines of Tasmania. Part II.—The Skeleton. *Ibid.* Vol. XLVII. 1910, pp. 411—454. Measurements of one skull in the Royal Museum, Brussels, are on p. 414.

(*g*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* Bd. XLII. 1910, pp. 175—227. Individual measurements of 30 skulls in the museum of the Royal College of Surgeons are given on pp. 224—227. Most of them had been measured previously by Flower or Barnard Davis.

* R. J. A. Berry and A. W. D. Robertson in their Preliminary Communication on Fifty-three Tasmanian Crania, Forty-two of which are now recorded for the first time, *Proceedings of the Royal Society of Victoria*, Vol. XXII. (New Series), 1909, pp. 47—58, say of the specimens rejected by Harper and Clarke: "We have no hesitation whatsoever in stating that all six crania are the crania of pure-blood Tasmanians" (p. 49). And again: "Every one presents over 90 per cent. of the features so characteristically found in the skull of the Tasmanian aboriginal" (p. 50). The measurements of Nos. 1 *a* and 2 *a* do not support such emphatic statements.

† Measurements of the six Oxford specimens taken by Dr Gabriel Farmer had been published previously in: H. Ling Roth: *The Aborigines of Tasmania, with a Chapter on the Osteology by J. G. Garson.* 2nd edition. Halifax (England), 1899. The skulls were re-measured by F. H. S. Knowles and B. W. Freire-Marreco, and their results differed markedly from Farmer's. The latter supposed that 5 were ♂ and 1 ♀, but Turner gives 2 ♂ and 4 ♀. The measurements given in Turner's paper are the only ones of these skulls which have been used.

(*h*) R. J. A. Berry, A. W. D. Robertson and L. W. G. Büchner: The Craniometry of the Tasmanian Aboriginal. *Journal of the Royal Anthropological Institute.* Vol. XLIV. 1914, pp. 122—126. Individual measurements of 52 skulls are given in Table I. Sexes are not given, but the numbers correspond to those of the sexed series for which tracings have been provided by Berry and Robertson (Dioptographic Tracings in Four Normae of Fifty-two Tasmanian Crania. *Transactions of the Royal Society of Victoria.* Vol. v. 1909). Nos. 1—14 had been measured previously by Harper and Clarke (reference (*b*) above) and they are preserved in the Hobart Museum. The others are in the same Museum, in the National Museum, Melbourne, in the Launceston Museum and in private collections. No. 48 was omitted as it is young adult and un-sexed. Several of the measurements are insufficiently defined. The "orbital intervals" (Measurement No. 41 = *DC*?) of skull 24 and later numbers are obviously incorrect.

(*i*) R. Pöch: Ein Tasmanierschädel im k. k. naturhistorischen Hofmuseum. Die anthropologische und ethnographische Stellung der Tasmanier. *Mitteilungen der anthropologischen Gesellschaft in Wien.* Bd. XLVI. 1916, pp. 37—91. Detailed measurements are given on pp. 41—44.

5. *A Comparison of Australian and Tasmanian Skulls.* The ♂ and ♀ means of the pooled Tasmanian skulls are given in Table VI together with those of the Australian *A* series. All the available and sufficiently defined measurements are included in that table, and many of the means are based on such small numbers of individuals that they can be of no permanent value by themselves. They are recorded for future use when measurements of other skulls may be added to them. Enough reliable means are given, however, to establish all the more salient features of the two types: such characters are alone considered in the following comparison between them. The sexual differences between direct measurements are of the same order as those usually found, though particularly large ones are given for the Australian *A*, *L* and *U*, owing, no doubt, to the unusual growth of the superciliary ridge on the ♂ skull. There is a satisfactory agreement between the ♂ and ♀ indices and angles. From a casual inspection, it is clear that the two racial types differ quite markedly. Using 27 characters and the Australian *A* standard deviations (given in Table VII below) a ♂ coefficient of racial likeness is found of $13{\cdot}54 \pm {\cdot}18$ between the Australian *A* (113·1) and Tasmanian (42·6) means. For nine indices it is $18{\cdot}59 \pm {\cdot}30$. The most significant differences are for 100 *B*/*F* ($\alpha = 72{\cdot}33$), $O_2$ (55·13), 100 *NB*/*NH*′ (30·18), 100 *B*/*H*′ (29·36), Flower's orbital index (29·10), *G*′*H* (25·84), *B* (25·79), *LB* (22·72), *NH*′ (22·71), and *F* (21·38), while *B*′, *C*, *J*, *NB*, Flower's $O_1$, *fml*, *fmb*, *U*, *S*, *H*′, Broca's $G_1$ and $G_2$, 100 *H*′/*F*, 100 *fmb*/*fml*, 100 *G*′*H*/*J*, 100 *GL*/*LB* and Broca's palatal index are hardly, if at all, distinguished. As fewer ♀ Australian *A* standard deviations are available, the ♀ coefficient could only be computed for 16 characters. Between the *A* type Australians (61·6) and Tasmanians (29·3) it is $7{\cdot}18 \pm {\cdot}23$. The differentiated measurements are 100 *B*/*F* ($\alpha = 43{\cdot}66$), *B* (30·72), 100 *B*/*H*′ (18·16) and 100 *NB*/*NH*′ (9·28), while *F*, *B*′, *C*, *LB*, *J*, *NB*, $O_2$, *fml*, *fmb*, *U*, *H*′ and 100 *fmb*/*fml* show differences

which are only just, if at all, significant*. It is somewhat disconcerting to find a most marked difference between the ♂ orbital heights ($O_2$) and no appreciable difference between the ♀ values. Male mean orbital indices are usually smaller than the ♀ means for the same race, but those sexual differences are decidedly greater for the Australian and Tasmanian than for European series. It may be supposed that the greater growth of the superciliary ridges gives the ♂ skull a proportionately smaller orbital height. For the Tasmanians the ♂ $O_2$ is actually less than the ♀ value, which is quite unusual, so for that type the growth of the frontal bone seems to have had a greater effect on the orbits than is the case with Australians or more advanced types.

A curious fact may be noticed: the Tasmanian ♂ calvarial breadth ($B$) is significantly greater than the ♂ Australian value, but for all other calvarial breadths (viz. $B'$, $B''$, bi-stephanic $B$, bi-asterionic $B$, external bi-orbital $B$ and internal bi-orbital $B$) the Australian mean exceeds the Tasmanian and the same is found for the bi-zygomatic, orbital and palato-maxillary, but not for the nasal breadth. A similar excess in the Tasmanian parietal breadth and of defect in other breadth measurements are shown by most of the ♀ means. From the above comparison it is clear that there is a marked distinction between the Australian and Tasmanian racial types. They belong, doubtless, to the same family of races, but the breach between them is much wider than one normally found between two contiguous populations. The greatest differences found between Australian types are very small compared with their differences from the Tasmanian type.

6. *The Variability of the Australian and Tasmanian Skulls.* The only enquiries concerning the variability of sexed series of Australian and Tasmanian skulls which have hitherto been published appear to be those of Robertson†, and of Berry, Robertson and Cross‡. Although purporting to use biometric methods, these studies fail to fulfil several of the conditions which the biometrician would consider essential. The Australian material consisted of 78 ♂ skulls and 22 ♀, so the one group is just large enough to furnish constants of variability, but the other is quite inadequate. No attempt was made to test whether the sample was racially homogeneous or not. The 54 ♂ and 30 ♀ Tasmanian crania are again insufficient and it may be doubted whether the constants of an unsexed series are of any value§. The only characters dealt with were $L$, $B$ and $H'$, and the resulting

* The writer is again tempted to suggest that the ♀ coefficient, and the individual $\alpha$'s on which it is based, are smaller than the corresponding ♂ values owing to the fact that the ♀ means are based on smaller numbers of skulls. A direct comparison of the means would certainly not suggest that the ♀ types are more similar to one another than the ♂ types for characters other than the orbital height.

† A. W. D. Robertson: Craniological Observations on the Lengths, Breadths and Heights of a Hundred Australian Aboriginal Crania. *Proceedings of the Royal Society of Edinburgh*, Session 1910—1911, Vol. xxxi. Part i. No. 1.

‡ R. J. A. Berry, A. W. D. Robertson and K. Stuart Cross: A Biometrical Study of the Relative Degree of Purity of Race of the Tasmanian, Australian, and Papuan. *Ibid.* No. 2.

§ Standard deviations of 53 characters for an unsexed series of 52 Tasmanian skulls have also been given by R. J. A. Berry, A. W. D. Robertson and L. W. G. Büchner: The Craniometry of the Tasmanian Aboriginal. *Journal of the Royal Anthropological Institute*, Vol. xliv. 1914, pp. 122—126.

| Character* | Males | | Females | |
|---|---|---|---|---|
| | Australian *A*† | Tasmanian | Australian *A*† | Tasmanian |
| *C* ... ... ... | 1294·6 (146) | 1264·3 (33) | 1147·4 (67) | 1153·8 (25) |
| *L* ... ... ... | 187·8 (82) | 182·2 (43) | 175·9 (38) | 174·6 (20) |
| *L′* ... ... ... | 185·8 (26) | — | 172·2 (6) | — |
| *F* ... ... ... | 186·4 (126) | 181·5 (49) | 176·1 (55) | 173·1 (26) |
| *B* ... ... ... | 132·2 (162) | 136·0 (60) | 127·3 (75) | 132·4 (36) |
| *B′* ... ... ... | 96·1 (138) | 94·0 (62) | 91·7 (58) | 90·1 (35) |
| *B″* ... ... ... | 110·1 (11) | 108·2 (24) | 106·6 (7) | 103·6 (10) |
| Bi-stephanic *B* ... | 104·5 (62) | 103·0 (22) | 102·5 (23) | 102·6 (14) |
| Bi-asterionic *B* ... | 108·5 (62) | 107·6 (40) | 103·4 (24) | 105·3 (21) |
| *H′* ... ... ... | 133·1 (144) | 130·9 (55) | 125·4 (71) | 125·3 (35) |
| *H* ... ... ... | 138·1 (17) | — | 130·2 (4) | — |
| *OH* ... ... ... | 115·0 (13) | 109·0 (1) | 110·0 (7) | — |
| *βOH* ... ... ... | 113·1 (10) | 108·0 (1) | 108·6 (5) | — |
| *LB* ... ... ... | 102·1 (137) | 98·8 (55) | 95·1 (66) | 92·7 (34) |
| *Q′* ... ... ... | 300·8 (13) | 300·0 (1) | 290·0 (6) | — |
| *βQ′* ... ... ... | 297·1 (27) | 295·0 (1) | 284·0 (12) | — |
| Broca's transverse arc | 294·5 (49) | 290·2 (40) | 276·1 (17) | 283·5 (17) |
| *S* ... ... ... | 373·0 (93) | 365·8 (36) | 353·4 (39) | 350·5 (15) |
| $S_1$ ... ... ... | 129·7 (77) | 127·2 (44) | 121·6 (25) | 121·3 (23) |
| $S_2$ ... ... ... | 129·7 (75) | 126·2 (42) | 121·9 (24) | 122·2 (23) |
| $S_3$ ... ... ... | 114·3 (72) | 111·8 (37) | 108·4 (22) | 109·4 (16) |
| Arc lambda to inion ... | 66·2 (40) | 68·6 (9) | 58·9 (10) | 65·0 (7) |
| Arc inion to opisthion | 49·1 (38) | 43·5 (8) | 47·9 (10) | 43·3 (7) |
| $S_1'$ ... ... ... | 113·3 (11) | 112·0 (1) | 108·2 (5) | — |
| $S_2'$ ... ... ... | 116·6 (10) | 108·0 (1) | 108·0 (4) | — |
| $S_3'$ ... ... ... | 97·4 (11) | 93·0 (1) | 91·0 (6) | — |
| Chord lambda to inion | 60·1 (9) | 62·0 (1) | 60·0 (4) | — |
| Chord inion to opisthion | 50·3 (9) | 47·0 (1) | 44·6 (5) | — |
| *U* ... ... ... | 515·6 (137) | 511·3 (48) | 488·8 (60) | 489·5 (23) |
| Glabella *U* ... ... | 530·2 (26) | 502·1 (11) | 495·7 (16) | 489·0 (5) |
| *PH* ... ... ... | 20·0 (1) | 17·3 (4) | — | 15·0 (4) |
| *G′H* ... ... ... | 66·8 (79) | 62·5 (36) | 62·1 (32) | 59·9 (16) |
| *GB* ... ... ... | 93·9 (37) | 88·3 (3) | 89·9 (9) | 91·0 (1) |
| *J* ... ... ... | 133·6 (139) | 131·0 (44) | 122·5 (65) | 122·0 (21) |
| Ext. bi-orbital *B* ... | 107·9 (40) | 105·4 (31) | 105·2 (10) | 101·5 (11) |
| Int. bi-orbital *B* ... | 100·6 (7) | 97·7 (6) | 98·4 (5) | 93·8 (4) |
| *NH′* ... ... ... | 49·5 (118) | 47·1 (58) | 45·6 (49) | 44·9 (30) |
| *NH*, *R* or *L* ... ... | 52·3 (3) | — | 44·0 (1) | — |
| *NB* ... ... ... | 26·9 (120) | 27·8 (57) | 25·5 (51) | 26·3 (29) |
| Ht. pyriform aperture | 31·0 (56) | 32·6 (10) | 28·9 (27) | 28·0 (13) |
| *DC* ... ... ... | 23·5 (2) | 22·4 (7) | 21·0 (3) | 23·6 (8) |
| Flower's inter-orb. *B*... | 26·2 (86) | 25·3 (20) | 24·7 (34) | 23·8 (13) |
| *SC* ... ... ... | 8·9 (8) | 7·0 (1) | 9·0 (5) | 7·5 (1) |

| Character* | Males | | Females | |
|---|---|---|---|---|
| | Australian *A*† | Tasmanian | Australian *A*† | Tasmanian |
| Flower's $O_1$ ... ... | 41·3 (83) | 41·1 (19) | 38·7 (35) | 38·8 (13) |
| $O_1'$ ... ... ... | 40·7 (33) | 39·3 (40) | 38·4 (17) | 38·3 (18) |
| $O_1$ ... ... ... | 43·3 (11) | 42·0 (1) | 40·3 (4) | — |
| $O_2$ ... ... ... | 33·5 (118) | 31·05 (60) | 32·9 (51) | 31·7 (31) |
| Palato-max. *L* ... ... | 60·4 (53) | 56·1 (37) | 54·7 (18) | 52·9 (14) |
| Broca's $G_1$ ... ... | 60·0 (54) | 60·3 (15) | 58·2 (26) | 55·1 (11) |
| $G_1'$ ... ... ... | 50·9 (18) | — | 50·1 (8) | — |
| Palato-max. *B* ... ... | 65·8 (55) | 64·8 (34) | 61·7 (18) | 61·7 (13) |
| Broca's $G_2$ ... ... | 38·2 (57) | 37·8 (17) | 36·2 (30) | 35·4 (14) |
| $G_2$ ... ... ... | 41·1 (28) | — | 39·1 (20) | — |
| *GL* ... ... ... | 103·2 (106) | 101·2 (50) | 97·4 (44) | 95·9 (29) |
| *fml* ... ... ... | 35·2 (130) | 35·7 (53) | 33·6 (65) | 34·2 (31) |
| *fmb* ... ... ... | 30·0 (111) | 29·6 (44) | 28·7 (54) | 28·4 (27) |
| Nasion-inion chord ... | 169·3 (28) | 175·0 (1) | 158·6 (18) | — |
| Glabella-inion chord ... | 177·0 (28) | 177·7 (36) | 164·9 (15) | 166·3 (16) |
| Sub. to gb.-inion chord | — | 99·7 (7) | — | 100·0 (2) |
| 100 *B*/*L* ... ... | 70·8 (94) | 74·2 (43) | 72·2 (47) | 75·1 (19) |
| 100 *B*/*F* ... ... | 71·0 (126) | 75·2 (49) | 72·4 (55) | 76·5 (26) |
| 100 *H′*/*L* ... ... | 71·8 (91) | 71·3 (37) | 71·6 (46) | 71·1 (19) |
| 100 *H′*/*F* ... ... | 71·6 (122) | 72·2 (44) | 70·9 (46) | 72·3 (26) |
| 100 *H*/*L* ... ... | 72·1 (1) | — | 69·8 (1) | — |
| 100 *B*/*H′* ... ... | 99·3 (156) | 103·9 (55) | 101·2 (79) | 105·7 (34) |
| 100 *B*/*H* ... ... | 96·6 (17) | — | 97·8 (4) | — |
| 100 (*B* − *H′*)/*L* ... ... | − 1·0 (90) | + 2·9 (37) | + 0·6 (46) | + 3·5 (19) |
| 100 *G′H*/*GB* ... ... | 72·2 (29) | 67·5 (2) | 69·0 (4) | — |
| 100 *NB*/*NH′* ... ... | 54·6 (132) | 59·1 (57) | 55·7 (58) | 59·0 (29) |
| 100 *NB*/*NH*, *R* or *L* ... | 49·2 (3) | — | 56·8 (1) | — |
| Flower's 100 $O_2/O_1$ ... | 82·0 (83) | 74·3 (19) | 84·1 (34) | 81·3 (13) |
| 100 $O_2/O_1'$ ... ... | 82·2 (45) | 79·4 (40) | 87·1 (25) | 83·3 (17) |
| 100 $O_2/O_1$ ... ... | 76·4 (11) | 64·3 (1) | 78·9 (5) | — |
| 100 *fmb*/*fml* ... ... | 85·0 (112) | 82·1 (42) | 85·0 (55) | 83·3 (26) |
| Palato-max. index ... | 110·0 (52) | 114·8 (34) | 112·9 (18) | 114·7 (12) |
| Broca's palatal index ... | 64·2 (54) | 62·8 (15) | 62·5 (26) | 64·6 (11) |
| 100 $G_2/G_1'$ ... ... | 81·8 (17) | — | 77·9 (8) | — |
| *Oc. I.* ... ... ... | 60·9 (9) | 57·7 (1) | 63·1 (4) | — |
| 100 *GL*/*LB* ... ... | 101·4 (101) | 103·1 (48) | 102·6 (47) | 102·8 (28) |
| 100 *G′H*/*J* ... ... | 50·2 (75) | 49·6 (23) | 51·1 (30) | 49·6 (7) |
| 100 *J*/*B* ... ... | 101·0 (139) | 95·9 (44) | 96·4 (65) | 92·1 (21) |
| Schwalbe's index ... | — | 55·1 (7) | — | 59·4 (2) |
| *P* ∠ ... ... ... | 84°·2 (14) | — | 79°·0 (2) | — |
| *N* ∠ ... ... ... | 72°·1 (44) | 73°·5 (32) | 71°·4 (13) | 73°·2 (14) |
| *A* ∠ ... ... ... | 70°·6 (44) | 69°·6 (32) | 71°·7 (13) | 70°·3 (14) |
| *B* ∠ ... ... ... | 37°·3 (44) | 36°·9 (32) | 36°·9 (13) | 36°·5 (14) |
| Foraminal angle‡ ... | − 4°·2 (9) | — | − 5°·0 (5) | — |

* For definitions of the measurements see pp. 417—418 above.

† The numbers of skulls available for the *A* group means are rather larger than the sums of the corresponding numbers given for the component groups (i.e. all in Table I except the Northern Territory) because there were a few individuals who did not come from the Northern Territory, but who could not be assigned to any one of the other territorial divisions.

‡ The foraminal angle is negative if the opisthion is lower than the basion with regard to the Frankfort horizontal plane.

TABLE VII. *Constants of Variation for Australian and Tasmanian Skulls.*

| Character | Males | | | | | | Females | | |
|---|---|---|---|---|---|---|---|---|---|
| | *A* Group Australians | | Tasmanians | | Egyptians *E*‡ | English§ | *A* Group Australians | | Egyptians *E*‡ |
| | Standard† Deviation | Coefficient† of Variation | Standard† Deviation | Coefficient† of Variation | Coefficient of Variation | Coefficient of Variation | Standard† Deviation | Coefficient† of Variation | Coefficient of Variation |
| *C* ... ... ... | 120·0 ±4·7 | 9·27± ·37 | — | — | 7·89± ·14 | 8·78± ·45 | 80·1 ±4·7 | 6·98± ·41 | 7·08± ·16 |
| *F* ... ... ... | 6·29± ·27 | 3·37± ·14 | — | — | 3·14± ·05 | 3·43± ·12 | 4·79± ·31 | 2·72± ·17 | 2·71± ·05 |
| *L* ... ... ... | 6·67± ·35 | 3·55± ·19 | — | — | 3·09± ·05 | 3·42± ·12 | — | — | 2·66± ·05 |
| *B* ... ... ... | 4·95± ·19 | 3·74± ·14 | 5·32± ·33 | 3·91± ·24 | 3·43± ·05 | 4·14± ·17 | 4·54± ·25 | 3·57± ·20 | 3·34± ·07 |
| *B′* ... ... ... | 5·36± ·22 | 5·58± ·23 | 4·81± ·29 | 5·12± ·31 | 4·28± ·07 | 4·73± ·18 | 3·69± ·23 | 4·02± ·25 | 4·11± ·08 |
| Bi-stephanic *B* ... ... | 8·81± ·53 | 8·43± ·51 | — | — | — | — | — | — | — |
| Bi-asterionic *B* ... ... | 5·69± ·34 | 5·24± ·32 | — | — | — | — | — | — | — |
| *H′* ... ... ... | 4·94± ·20 | 3·71± ·15 | 4·76± ·31 | 3·64± ·23 | — | 3·90± ·17 | 5·30± ·30 | 4·23± ·24 | — |
| *H* ... ... ... | — | — | — | — | 3·75± ·06 | 4·09± ·23 | — | — | 3·39± ·07 |
| *LB* ... ... ... | 4·34± ·18 | 4·25± ·17 | 4·13± ·27 | 4·18± ·27 | 3·90± ·06 | 4·47± ·20 | 4·16± ·24 | 4·37± ·26 | 3·65± ·07 |
| *U* ... ... ... | 16·50± ·67 | 3·20± ·13 | — | — | 2·65± ·04 | 3·00± ·13 | 11·05± ·68 | 2·26± ·14 | 2·35± ·05 |
| *S* ... ... ... | 13·62± ·67 | 3·65± ·18 | — | — | 3·36± ·05 | 3·75± ·16 | — | — | 2·92± ·06 |
| $S_1$ ... ... ... | 6·81± ·37 | 5·25± ·29 | — | — | 4·88± ·08 | 5·00± ·19 | — | — | 4·56± ·09 |
| $S_2$ ... ... ... | 7·43± ·41 | 5·73± ·32 | — | — | 5·77± ·09 | 6·24± ·25 | — | — | 5·19± ·10 |
| $S_3$ ... ... ... | 6·17± ·35 | 5·40± ·30 | — | — | 5·91± ·10 | 6·50± ·27 | — | — | 6·06± ·12 |
| *G′H* ... ... ... | 4·21± ·23 | 6·30± ·34 | — | — | 5·90± ·10 | 6·31± ·33 | — | — | 5·64± ·11 |
| *J* ... ... ... | 6·07± ·25 | 4·54± ·18 | — | — | 3·55± ·06 | 3·69± ·27 | 5·84± ·35 | 4·77± ·28 | 3·62± ·08 |
| *NH′* ... ... ... | 3·14± ·14 | 6·34± ·28 | 3·47± ·22 | 7·37± ·46 | — | 6·75± ·45 | — | — | — |
| *NH* ... ... ... | — | — | — | — | 5·65± ·09 | 6·03± ·31* | — | — | 5·31± ·10 |
| *NB* ... ... ... | 2·00± ·07 | 7·43± ·33 | 1·69± ·11 | 6·06± ·38 | 7·27± ·12 | 8·17± ·43 | 1·89± ·13 | 7·41± ·50 | 6·98± ·14 |
| Ht. pyriform aperture ... | 2·91± ·19 | 9·39± ·60 | — | — | — | — | — | — | — |
| Flower's $O_1$ ... ... | 2·56± ·13 | 6·20± ·33 | — | — | — | — | — | — | — |
| $O_1'$ ... ... ... | — | — | — | — | — | 4·20± ·22 | — | — | — |
| $O_2$ ... ... ... | 2·08± ·09 | 6·21± ·27 | 2·19± ·13 | 6·70± ·41 | 5·56± ·09* | 6·70± ·34* | 2·14± ·14 | 6·50± ·43 | 5·62± ·11 |
| Flower's inter-orb. *B* ... | 2·06± ·11 | 7·86± ·41 | — | — | — | — | — | — | — |
| *DC* ... ... ... | — | — | — | — | — | 9·73± ·51 | — | — | — |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Broca's $G_1$ ... ... | 4·54± ·29 | 7·57± ·49 | — | — | — | — | — | — | — | — |
| $G_1$ ... ... ... | — | — | — | — | 6·70± ·11 | 5·59± ·33 | — | — | 6·26± ·13 | — |
| Palato-max. $L$ ... ... | 3·93± ·26 | 6·51± ·43 | — | — | — | — | — | — | — | — |
| Broca's $G_2$ ... ... | 2·93± ·19 | 7·67± ·49 | — | — | — | — | — | — | — | — |
| $G_2$ ... ... ... | — | — | — | — | 6·78± ·12 | 7·76± ·51 | — | — | 6·83± ·15 | — |
| Palato-max. $B$ ... ... | 3·31± ·21 | 5·03± ·32 | — | — | — | — | — | — | — | — |
| $GL$ ... ... ... | 5·88± ·27 | 5·70± ·37 | 5·40± ·36 | 5·34± ·36 | 5·10± ·08 | 5·66± ·34 | — | — | 4·49± ·09 | — |
| $fml$ ... ... ... | 2·52± ·11 | 7·16± ·30 | 2·82± ·18 | 7·90± ·52 | 6·95± ·11 | 8·15± ·36 | 2·28± ·13 | 6·79± ·40 | 6·37± ·12 | — |
| $fmb$ ... ... ... | 2·17± ·10 | 7·23± ·33 | — | — | 7·18± ·11 | 7·19± ·33 | 2·25± ·15 | 7·84± ·51 | 6·96± ·14 | — |
| $100\,B/L$ ... ... | 3·88± ·19 | 5·48± ·27 | — | — | 3·57± ·06 | 4·62± ·19 | — | — | 3·34± ·07 | — |
| $100\,B/F$ ... ... | 2·89± ·12 | 4·07± ·17 | — | — | — | — | 2·48± ·16 | 3·43± ·22 | — | — |
| $100\,H'/L$ ... ... | 3·70± ·19 | 5·15± ·26 | — | — | — | 4·72± ·21 | — | — | — | — |
| $100\,H'/F$ ... ... | 3·09± ·13 | 4·32± ·19 | — | — | — | — | — | — | — | — |
| $100\,H/L$ ... ... | — | — | — | — | 4·06± ·07 | 4·65± ·26 | — | — | 3·90± ·08 | — |
| $100\,B/H'$ ... ... | 5·41± ·21 | 5·45± ·21 | 4·05± ·26 | 3·90± ·25 | — | 4·80± ·21 | 5·15± ·28 | 5·09± ·27 | — | — |
| $100\,B/H$ ... ... | — | — | — | — | 4·14± ·07 | 4·35± ·24 | — | — | 3·88± ·08 | — |
| $100\,NB/NH'$ ... ... | 5·17± ·21 | 9·47± ·40 | 4·75± ·30 | 8·04± ·51 | — | — | 4·76± ·30 | 8·55± ·54 | — | — |
| $100\,NB/NH$ ... ... | — | — | — | — | 8·08± ·13 | 9·22± ·49* | — | — | 7·90± ·16 | — |
| Flower's orb. index ... | 5·61± ·29 | 6·84± ·36 | — | — | — | — | — | — | — | — |
| $100\,O_2/O_1'$ ... ... | — | — | — | — | — | 7·45± ·40 | — | — | — | — |
| Broca's palatal index ... | 6·98± ·45 | 10·87± ·71 | — | — | — | — | — | — | — | — |
| $100\,G_2/G_1$ ... ... | — | — | — | — | 8·67± ·16 | 7·04± ·50 | — | — | 8·40± ·19 | — |
| Palato-max. index ... | 7·74± ·51 | 7·04± ·47 | — | — | — | — | — | — | — | — |
| $100\,fmb/fml$ ... ... | 5·83± ·26 | 6·86± ·31 | — | — | 6·86± ·11 | 7·07± ·33 | 6·50± ·42 | 7·65± ·49 | 6·69± ·13 | — |
| $100\,J/B$ ... ... | 4·59± ·19 | 4·54± ·18 | — | — | — | — | 4·32± ·26 | 4·48± ·27 | — | — |
| $100\,G'H/J$ ... ... | 3·45± ·19 | 6·87± ·38 | — | — | — | — | — | — | — | — |
| $100\,GL/LB$ ... ... | 4·12± ·20 | 4·06± ·19 | — | — | — | — | — | — | — | — |

* These measurements were taken on the left side.

† The numbers of skulls on which the Australian *A* and Tasmanian constants of variation were calculated are precisely the same as the corresponding ones for the means given in Table VI.

‡ Quoted from: Karl Pearson and Adelaide G. Davin: On the Biometric Constants of the Human Skull. *Biometrika*, Vol. xvi. 1924, pp. 338—9.

§ Quoted from: Beatrix G. E. Hooke: A Third Study of the English Skull with special reference to the Farringdon Street Crania. *Ibid.* Vol. xviii. 1926, p. 40.

indices 100 $B/L$, 100 $H'/L$ and 100 $H'/B$, and it is not to be expected that any reliable determination of the variability of the skull as a whole will be given by considering only six correlated measurements. In dealing with the Tasmanian skulls, Flower's ophryo-occipital lengths ($F$) were pooled with others measured from the glabella.

Standard deviations and coefficients of variation of the Australian $A$ ♂ and ♀ and the Tasmanian ♂ series dealt with in the present paper are in Table VII. Constants are given for all characters for which 50 or more skulls had been measured, and none quoted are based on fewer than that number of individuals. The Australian skulls of the $A$ type were drawn from such an enormous area that they may be expected to show a high variability. An unusually great variation might be an inherent characteristic of the race, or it might be occasioned by the conditions to which it had been subjected, or again, it might be due to racial heterogeneity. The personal equations of the different workers who measured the series may also have influenced the observed variation to some extent. By dividing the population into groups (Section 3 above) the conclusion was reached that the skulls from all parts of the continent, except the Northern Territory, belonged to a single race. That assumption has been made in spite of the fact that a significant, though slight, difference was found between a group comprising crania from Western Australia, South Australia and Victoria, and another representing Queensland and New South Wales. A further test of the validity of that assumption will be given

TABLE VIII.

*Standard Deviations of Groups of Australian Male Skulls.*

| | | $C$ | $F$ | $B$ | $B'$ | $H'$ | $LB$ | $U$ | $J$ | $fm$ |
|---|---|---|---|---|---|---|---|---|---|---|
| W. Australia, S. Australia and Victoria | $\sigma$ | 118·1 ± 6·4 | 6·36 ± ·35 | 4·75 ± ·19 | 5·30 ± ·30 | 4·90 ± ·26 | 4·32 ± ·23 | 17·42 ± 1·01 | 4·69 ± ·26 | 2·12 |
| | No. of Skulls | 77 | 74 | 87 | 69 | 83 | 79 | 68 | 72 | 6 |
| Australian $A$ | $\sigma$ | 120·0 ± 4·7 | 6·29 ± ·27 | 4·95 ± ·19 | 5·36 ± ·22 | 4·94 ± ·20 | 4·34 ± ·18 | 16·50 ± ·67 | 6·07 ± ·25 | 2·17 |
| | No. of Skulls | 146 | 126 | 162 | 138 | 144 | 137 | 137 | 139 | 11 |

| | | $NH'$ | $NB$ | $O_2$ | $fml$ | 100 $B/F$ | 100 $H'/F$ | 100 $B/H'$ | 100 $NB/NH'$ | 100 $fm$ |
|---|---|---|---|---|---|---|---|---|---|---|
| W. Australia, S. Australia and Victoria | $\sigma$ | 2·83 ± ·16 | 2·07 ± ·12 | 2·07 ± ·12 | 2·57 ± ·14 | 3·06 ± ·17 | 3·57 ± ·20 | 5·27 ± ·27 | 5·00 ± ·28 | 5·51 |
| | No. of Skulls | 68 | 68 | 63 | 73 | 75 | 72 | 85 | 71 | 6 |
| Australian $A$ | $\sigma$ | 3·14 ± ·14 | 2·00 ± ·07 | 2·08 ± ·09 | 2·52 ± ·11 | 2·89 ± ·12 | 3·09 ± ·13 | 5·41 ± ·21 | 5·17 ± ·21 | 5·83 |
| | No. of Skulls | 118 | 120 | 118 | 130 | 126 | 122 | 156 | 132 | 11 |

by comparing the variability of the former group with the variability of the *A* group as a whole. Male standard deviations for 18 characters are given in Table VIII.

It will be seen that the variabilities of the combined Western Australia, South Australia and Victoria series are almost identically the same as those of the larger *A* group, of which they form approximately one-half. Of the 18 characters compared the standard deviations of 12 are increased by the addition of the Queensland and New South Wales skulls, the other six are reduced. When the $\sigma$'s of the two independent sub-samples are compared directly, it is found that the Western Australia, South Australia and Victoria series is more variable than that from Queensland and New South Wales for 11 characters and less variable for 7. Only 3 of the 18 differences are greater than 2·5 times their probable errors. For *J* the ratio is 3·6, for $100\,H'/F$, 3·1, and for $NH'$, 2·57. Judging from all the characters, it is reasonable to conclude that the Western Australia, South Australia and Victoria series is not appreciably less variable than the total population of *A* type skulls.

From the constants given in Table VII it should be possible to obtain an accurate idea of the variabilities of the Australian *A* and Tasmanian types relative to one another and to other racial populations. The table includes the coefficients of variation of the long Egyptian *E* series of 26th—30th Dynasty skulls, provided by Pearson and Davin, and of a London series from a 17th century cemetery in Farringdon Street, measured by Hooke. The former is remarkable on account of its small variation, and the constants of the latter may be considered typical for a racially homogeneous European population. The *A* group Australians have 7 ♂ coefficients of variation greater than the corresponding Tasmanian ones and 4 less, so that the population which is scattered over the larger area appears to be rather more variable than the restricted island one. Both are decidedly more variable than the Egyptian series. The Australians only have a lesser variation than the Egyptians for the characters $S_8$ and $S_3$ and the Tasmanians only for *NB*. The English coefficients of variation exceed the Australian for 14 out of 24 characters and the Tasmanian for 7 out of 10 characters. Finally, it may be noticed that the English coefficients exceed the Egyptian for 26 out of 28 characters. Arranging the types in order of their variability, as judged by all the cranial measurements available, we have the increasing sequence: Egyptian—Tasmanian—Australian *A*—English. The population which extends over the greater part of a continent is less variable than a collection of individuals from a single London cemetery. From the coefficients of variation, the relative variation of the ♂ Australian skull is seen to be greater than that of the ♀ skull, but the excess of one over the other is not so evident as is that of the ♂ over the ♀ variation in the case of the Egyptians. The ♀ Australian constants exceed the corresponding Egyptian ones in 9 cases out of 12.

When computing coefficients of racial likeness between groups of Australian skulls (Section 3 above) the Egyptian *E* standard deviations were used. If they were replaced by the more appropriate Australian *A* values which are now available, all the coefficients would be slightly reduced. Only one was re-calculated.

Between the Western Australia, South Australia and Victoria ♂ group and the Queensland and New South Wales ♂ group a coefficient was found of 1·58 ± ·21, using Egyptian standard deviations (Table IV). For the same 19 pairs of means and Australian *A* standard deviations the coefficient is 1·25 ± ·21.

7. *Additional Measurements of Australian and Tasmanian Skulls.* It might be thought that the foregoing analysis of the measurements of Australian and Tasmanian skulls provided by a considerable number of different workers would be sufficient, in itself, to establish the main characters of the craniological types. The evidence appeared to be perfectly consistent. After excluding a considerable part of the material, there was good reason to believe that the remainder had been dealt with by strictly comparable methods of technique. We are able to test the reliability of the results by comparing the pooled means with constants given for series not hitherto considered. The majority of these are unsexed series.

Of the Australian skulls in Paris museums, 10 only have been hitherto used. They are represented by a few means of 7 ♂ and 3 ♀ specimens given by Pruner-Bey in his paper of 1865. As the localities from which they had been obtained were not stated, the measurements could only be used in compiling the groups from unknown localities given in Table III above. In 1879 Broca provided the means of a series of 27 unsexed Australian skulls to which he had access in Paris*. In the same paper there are means of 35 unsexed Tasmanian skulls. Of that series, Klaatsch's unusual measurements of four specimens (see p. 426 (*d*)) and others of 1 ♀ given in the *Crania Ethnica* have alone been used in the present paper. Comparison is made with the pooled data in Table IX. The Australian skulls in Paris may have included a few from the Northern Territory, but there can be little doubt that the majority belong to the *A* group. Cauvin's last paper on Australian skulls was published in 1883†. It provides the individual cephalic, nasal and orbital indices of 50 unsexed skulls, one having come from the Northern Territory and the remainder from other known parts of the continent. Of that number, two were in the *Musée Broca*, and it is probable that Broca himself had measured them before 1879, but the others appear to have been in museums or private collections in Australia. Individual indices of 23 of the series had been given by Cauvin at an earlier date (see p. 419 above) and as they were sexed it was possible to use them when compiling the Australian *A* means. As they could not be separated from the others, all 49 were pooled to give the values in Table IX. There is a most satisfactory agreement between the groups represented in that table. All the absolute measurements of Broca's unsexed series are intermediate between the corresponding ♂ and ♀ means (with the unimportant exceptions of the Tasmanian *B'*, bi-stephanic *B* and $O_1'$) and they incline more towards the ♂ than the ♀ means.

* Paul Broca: Méthode des moyennes. Etude des variations craniométriques et de leur influence sur les moyennes; détermination de la série suffisante. *Bulletins de la Société d'Anthropologie de Paris*, Tome II, 3ième série, 1879, pp. 756—820.

† Cauvin: Sur les races de l'Océanie (Analyse). *Bulletins de la Société d'Anthropologie de Paris*, Tome VI, 3ième série, 1883, pp. 245—256. The reference to Cauvin's earlier paper is given on p. 419 above.

## TABLE IX.

*Means of Sexed and Unsexed Series of Australian and Tasmanian Skulls.*

| Character | Australians | | | | Tasmanians | | |
|---|---|---|---|---|---|---|---|
| | Pooled Australian *A*. Male | Broca's unsexed series | Cauvin's unsexed series | Pooled Australian *A*. Female | Pooled Tasmanians. Male | Broca's unsexed series | Pooled Tasmanians. Female |
| *L* ... ... ... | 187·9 (82) | 182·4 (27) | — | 175·9 (38) | 182·2 (43) | 180·4 (35) | 174·6 (20) |
| *B* ... ... ... | 132·2 (162) | 130·4 (27) | — | 127·3 (75) | 136·0 (60) | 135·5 (35) | 132·4 (36) |
| $B'_4$ ... ... ... | 96·1 (138) | 93·7 (27) | — | 91·7 (58) | 94·0 (62) | 94·0 (35) | 90·1 (35) |
| Bi-stephanic *B* ... | 104·5 (62) | 102·7 (27) | — | 102·5 (23) | 103·0 (22) | 104·8 (35) | 102·6 (14) |
| *H'* ... ... | 133·1 (144) | 131·8 (27) | — | 125·4 (71) | 130·9 (55) | 128·5 (35) | 125·3 (35) |
| *LB* ... ... | 102·1 (137) | 98·0 (27) | — | 95·1 (66) | 98·8 (55) | 96·1 (35) | 92·7 (34) |
| Ext. bi-orbital *B* | 107·9 (40) | 106·8 (27) | — | 105·2 (10) | 105·4 (31) | 104·5 (35) | 101·5 (11) |
| *J* ... ... ... | 133·6 (139) | 127·3 (27) | — | 122·5 (65) | 131·0 (44) | 125·7 (35) | 122·0 (21) |
| *NH'* ... ... | 49·5 (118) | 47·0 (27) | — | 45·6 (49) | 47·1 (58) | 45·6 (35) | 44·9 (30) |
| *NB* ... ... | 26·9 (120) | 26·1 (27) | — | 25·5 (51) | 27·8 (57) | 26·6 (35) | 26·3 (29) |
| $O_1'$ ... ... | 40·7 (33) | 40·4 (27) | — | 38·4 (17) | 39·3 (40) | 37·7 (35) | 38·3 (18) |
| $100\,B/L$ ... ... | 70·8 (94) | {71·5 (27)}[1] | 72·2 (48) | 72·2 (47) | 74·2 (43) | {75·1 (35)} | 75·1 (19) |
| $100\,H'/L$ ... ... | 71·8 (91) | {72·2 (27)} | — | 71·6 (46) | 71·3 (37) | {71·2 (35)} | 71·1 (19) |
| $100\,B/H'$ ... ... | 99·3 (156) | {98·9 (27)} | — | 101·2 (79) | 103·9 (55) | {105·5 (35)} | 105·7 (34) |
| $100\,NB/NH'$ ... | 54·6 (132) | {55·6 (27)} | 56·0 (49) | 55·7 (58) | 59·1 (57) | {58·4 (35)} | 59·0 (29) |
| $100\,O_2/O_1'$ ... | 82·2 (45) | — | 83·0 (49) | 87·1 (25) | 79·4 (40) | — | 83·3 (17) |

[1] The indices in curled brackets were calculated from the means of the component lengths instead of from individual indices.

The differences between the indices are statistically insignificant. This welcome evidence encourages us to place greater confidence in the Australian *A* and Tasmanian constants for the two sexes which have already been determined.

In 1897 Krause published individual measurements of 170 native skulls which were preserved at that time in various museums and private collections in Australia*. The district from which each had come was known in the case of all except four of them and of the remainder 21 were from the Northern Territory. The sex of 21 ♂ and 15 ♀ individuals was known with certainty, but unfortunately no attempt was made to sex the others†. Measurements were taken in accordance with the instructions of the *Frankfurter Verständigung*. Excluding the juvenile specimens and all from the Northern Territory and unknown localities, we find the unsexed means given in Table X. In 1910 Robertson dealt with 100 Australian skulls in the Museum of the Anatomical Department of the University of Melbourne‡; 96 were from States other than the Northern Territory and the provenance of the other four was not known. The number of measurements taken

* Wilhelm Krause: Australische Schädel. *Zeitschrift für Ethnologie. Verhandlungen der Berliner Gesellschaft für Anthropologie, Ethnologie und Urgeschichte*, Jahrgang 1897, pp. 508—558.

† On p. 512 Krause gives a list of 16 of the unsexed skulls which were ♀ in his opinion, but he made no attempt to distinguish the specimens which were ♀?.

‡ A. W. D. Robertson: Craniological Observations on the Lengths, Breadths and Heights of a Hundred Australian Aboriginal Crania. *Proceedings of the Royal Society of Edinburgh*, Session 1910—1911, Vol. xxxi. Part i. No. 1.

(viz. *L*, *B* and *H'* and three resulting indices) was too small to justify any of the wider generalisations which are broached in the paper. The lengths were taken in accordance with the instructions of the Monaco International Conference of 1906. Of the 100 skulls, 78 were supposed ♂ and 22 ♀, so there is an unusual preponderance of the one sex over the other. That might be explained by the fact that they were selected from a larger number of specimens, but it is not stated whether that was so or not. It is probable that all, or nearly all, of the 23 skulls described by Cauvin, which we have used in computing the Australian *A* pooled means, were re-measured by Krause and some of them may have been included in Robertson's material. Some others of Robertson's series may also have been measured previously by Krause. It is not possible to decide such points from the published writings of these authors. Turning to Table X, it will be seen that there is an excellent agreement between the indices of Robertson's series and the pooled Australian *A* means, but the absolute measurements show differences that are clearly significant. The ♂ *L*, *B* and *H'* of the Melbourne series are all decidedly smaller than the pooled ♂ means and the sexual differences for those characters are also smaller and most markedly so in the case of *B*. It is only possible to reconcile this evidence by supposing that part of the material compared had been incorrectly sexed. The assumption that some of the skulls in the Melbourne Museum which were judged ♂ are in reality ♀ would lead to a satisfactory agreement. That seems to be the most reasonable hypothesis: a transformation of that kind would give more usual sexual differences and the sex ratio would be more in accordance with that which might be expected for a museum collection.

The measurements of Krause's unsexed series are only given (Table X) for the characters for which reliable pooled means are available. Of the 15 absolute measurements, 9 are intermediate between the ♂ and ♀ values of the *A* group while they are nearer to the ♂ than to the ♀ mean. One other (*J*) is also intermediate, but it is rather nearer to the ♀ mean. Four other characters (*NB*, $O_2$, *fml* and *fmb*) have unsexed means 0·1 mm. greater than the corresponding ♂ values. But the calvarial breadth of Krause's series is decidedly smaller than the ♀ value for the pooled group. All the indices and angles agree well except the three involving *B* (viz. 100 *B*/*L*, 100 *B*/*H'* and 100 *J*/*B*) and the differences for the latter are perfectly significant in virtue of the considerable numbers on which the means are based. The peculiarly small value of *B* is alone responsible for the discordance of Krause's series and it is not associated with small values of such related characters as *B'*, *U* and *J*. This comparison suggests that the calvarial breadths used in computing the pooled mean and Krause's calvarial breadth had not been determined in the same way. If that could be demonstrated, then it might be possible to reconcile the two series, but no difference can be found between the methods by which the calvarial breadths were said to have been taken. Before attempting to find a way out of this *impasse* some further evidence may be considered. In compiling the pooled Australian means given in Table VI a few sources were overlooked. The following papers contain further measurements of ♂ adult skulls from localities other than the Northern Territory:

TABLE X.

*Further Mean Measurements of Series of Australian Skulls.*

| Character | ♂ | | | ♂+♀ | | ♀ | |
|---|---|---|---|---|---|---|---|
| | Pooled *A* Group: 1st Sample | Pooled *A* Group: 2nd Sample | Robertson's Series | Krause's Series | Robertson's Series | Pooled *A* Group | Robertson's Series |
| *L* ... | 187·8 (82) | 185·8 (28) | 183·6 (78) | 184·6 (135) | 181·8 (100) | 175·9 (38) | 175·6 (22) |
| *B* ... | 132·2 (162) | 133·7 (14) | 130·6 (78) | 124 8 (135) | 130·2 (100) | 127·3 (75) | 128·7 (22) |
| *B'* ... | 96·1 (138) | 96·2 (10) | — | 95·8 (134) | — | 91·7 (58) | — |
| *H'* ... | 133·1 (144) | 137·8 (6) | 131·1 (78) | 132·1 (95) | 129·7 (100) | 125·4 (71) | 124·5 (22) |
| *LB* ... | 102·1 (137) | 101·4 (28) | — | 99·6 (128) | — | 95·1 (66) | — |
| *S* ... | 373·0 (93) | 376·5 (13) | — | 368·6 (128) | — | 353·4 (39) | — |
| *U* ... | 515·6 (137) | 514·5 (13) | — | 502·6 (135) | — | 488·8 (60) | — |
| *J* ... | 133·6 (139) | 134·6 (14) | — | 127·4 (100) | — | 122·5 (65) | — |
| *G'H* ... | 66·8 (79) | 68·7 (13) | — | 65·4 (122) | — | 62·1 (32) | — |
| *NB* ... | 26·9 (120) | 27·9 (23) | — | 27·0 (128) | — | 25·5 (51) | — |
| *GL* ... | 103·2 (106) | 101·1 (22) | — | 100·4 (97) | — | 97·4 (44) | — |
| *fml* ... | 35·2 (132) | 35·5 (24) | — | 35·3 (125) | — | 33·6 (65) | — |
| *fmb* ... | 30·0 (111) | 30·0 (23) | — | 30·1 (124) | — | 28·7 (54) | — |
| $O_2$ ... | 33·5 (118) | 33·7 (23) | — | 33·6 (105) | — | 32·9 (51) | — |
| *C* ... | 1294·6 (146) | 1282·1 (14) | — | 1242·8 (47) | — | 1147·4 (67) | — |
| 100 *B*/*L* ... | 70·8 (94) | 71·0 (28) | 71·3 (38) | {67·6 (135)} | 71·8 (100) | 72·2 (47) | 73·4 (22) |
| 100 *H'*/*L* ... | 71·8 (91) | 72·1 (5) | 71·5 (78) | {71·6 (95)} | 71·4 (100) | 71·6 (46) | 71·0 (22) |
| 100 *B*/*H'* ... | 99·3 (156) | 97·8 (6) | {99·6 (78)}[1] | {94·5 (95)} | {100·4 (100)} | 101·2 (79) | {103·4 (22)} |
| 100 *J*/*B* ... | 101·0 (139) | 100·6 (14) | — | {102·0 (100)} | — | 96·4 (65) | — |
| 100 *fmb*/*fml* | 85·0 (112) | 84·2 (23) | — | {85·3 (124)} | — | 85·0 (55) | — |
| 100 *G'H*/*J*... | 50·2 (75) | 51·1 (13) | — | {51·4 (100)} | — | 51·1 (30) | — |
| 100 *GL*/*LB* | 101·4 (101) | 100·0 (22) | — | {100·8 (97)} | — | 102·6 (47) | — |
| *N* ∠ ... | 72°·1 (44) | 70°·2 (21) | — | {71°·4 (97)} | — | 71°·4 (13) | — |
| *A* ∠ ... | 70°·6 (44) | 71°·0 (21) | — | {70°·4 (97)} | — | 71°·7 (13) | — |
| *B* ∠ ... | 37°·3 (44) | 38°·8 (21) | — | {38°·2 (97)} | — | 36°·9 (13) | — |

[1] The indices in curled brackets were calculated from the means of the component lengths.

(*a*) K. Brackebusch: *Die Australierschädel der Sammlung des anatomischen Instituts.* Inaugural-Dissertation. Göttingen, 1905. This paper gives a large number of measurements of 19 ♂ skulls. Of that number 16 figure in Spengel's catalogue of 1874 (reference (*b*), p. 419), but additional measurements given by Brackebusch can now be used. All of No. XVI may be included as the measurements of this skull were only used previously in the group from unknown localities.

(*b*) J. T. Wilson: Craniometry. Abstract Report on the Craniology of Australian Aborigines, pp. 96—100 in John Fraser: *The Aborigines of New South Wales.* Published by authority of the New South Wales Commissioners for the World's Columbian Exposition, Chicago, 1893. Measurements of 4 ♂ skulls in the Australian Museum, Sydney, are given. They may have been measured previously by Cauvin and subsequently by Krause.

(*c*) A. Ecker: *Die anthropologischen Sammlungen Deutschlands.* Freiburg section, 1878. Nos. 2, 3 and 5 (p. 53) had been omitted in error when compiling

the first series of pooled means. No. 11 (p. 54) was only included then in the group from unknown localities, so it may be re-used now.

(*d*) H. Basedow: Der Tasmanierschädel, ein Insulartypus. *Zeitschrift für Ethnologie.* 42 Jahrgang, 1910, pp. 176—227. No. 29 (pp. 212—215) was only included previously in the group from unknown localities.

(*e*) *Craniological Data from the Indian Museum, Calcutta.* Ethnographic Survey of India. Calcutta, 1909. Measurements of one skull are given on p. 52.

This small group of additional material leads to the ♂ means (called the Pooled *A* Group: 2nd Sample) given in column 3 of Table X. It will be seen that they are in very close agreement with those previously found. Using 15 characters for which Australian *A* standard deviations are available (viz. all given in Table X except *H'*, *S*, *GL*, 100 *H'/L*, 100 *B/H'*, 100 *J/B*, 100 *G'H/J*, *N*∠, *A*∠ and *B*∠) the coefficient of racial likeness between the 1st (120·4) and 2nd (20·0) samples is $-0{\cdot}05 \pm {\cdot}24$. The mean *B* of the 2nd sample actually exceeds that of the 1st, though by an insignificant amount.

Professor Wilhelm Krause was one of the German anthropologists who ratified the craniometric scheme known as the *Frankfurter Verständigung**. In measuring the Australian skulls he followed that system and no mention whatever is made of a deviation from it†. The only calvarial breadth defined is the "Grösste Breite: senkrecht zur Sagittalebene, gemessen mit dem Schiebezirkel, wo sie sich findet, nur nicht am Zitzenfortsatz, Processus mastoides, oder an der hinteren Temporalleiste; die Messpunkte müssen in derselben Horizontalebene liegen" (*loc. cit.* p. 2, No. 4). That greatest calvarial breadth measured on either the parietal bones or the temporal squamae is the one defined by all the above cited craniometricians with two exceptions. Flower determined the greatest parietal breadth only and he was followed by Basedow. Their mean *B* for 60 ♂ skulls of the Australian *A* type is 132·4 and when the parieto-squamous breadths of 102 other ♂ Australian *A* skulls is added to that number the mean is 132·2. It is clear that, for the statistically small numbers dealt with, the pooling of the breadths determined by the two methods will be quite a safe procedure and that was done in compiling the means given in Table VI. It may be inferred that the greatest calvarial breadth of the Australian skull is found almost invariably on the parietal bones, as normally for modern racial types‡. Strangely enough Krause failed to notice the discordance between his measurements for the race and others previously given. He gives a mean cephalic index of 69·7 for 155 unsexed skulls, but that is calculated for the glabella horizontal length (*L'*) which is, for ♂ mean values, several millimetres less than the greatest glabella-occipital length (*L*) according to his own measurements. Using the latter, and excluding the Northern Territory skulls

* Verständigung über ein gemeinsames craniometrisches Verfahren. *Archiv für Anthropologie*, Band xv. 1884, pp. 1—8.

† Krause also followed the *Frankfurter Verständigung* without modification in the third part of the Berlin section of *Die anthropologischen Sammlungen Deutschlands*, 1898.

‡ Pöch (*loc. cit.* p. 72) found for adult ♂ Australian skulls that the greatest breadth fell on the parietal bones in 19 cases out of 21.

and a few immature specimens, the mean of the usual cephalic index (100 $B/L$) is 67·6 for 135 unsexed skulls. Brackebusch (*loc. cit.* p. 14) and others have wrongly compared Krause's value of 69·7 with the mean cephalic indices given for other series of Australian skulls and have hence failed to notice the wide discrepancy between the records. On the one hand we have 19 anthropologists providing measurements of over 300 skulls which are in close agreement when comparison is restricted to those which are said to have been taken by precisely similar methods of technique; on the other hand a single anthropologist has given measurements which agree perfectly well with those of the other workers except in the case of one chord. It is extremely probable that the discordance does not indicate a racial difference between the series, but a difference between the ways in which that chord was measured. The most reasonable explanation seems to be that Krause did not measure the parieto-squamous but some other calvarial breadth*.

8. *Conclusions.* The main purpose of this paper has been to place on record some reliable statistical constants for the Australian and Tasmanian skulls. It appears that only two racial types can be found among the natives of the whole continent of Australia. The first is probably confined to the Northern Territory and it differs distinctly from the second—the *A* type, which is spread over all the remaining parts—by having a smaller brain-box and slightly differing proportions of height to length and breadth. No differences can be found between the facial skeletons of the two. The Australian races are as similar to one another as contiguous races are usually found to be. The Northern Territory type is not represented by a large enough sample to furnish statistical constants of any permanent value. The *A* type is to-day represented by a small population which is spread homogeneously, as far as we can tell, over an enormous area. A similar wide extension of a single racial variety is found in China for a large and dense population†, and the condition is extremely dissimilar to that which is found in Europe. The Australian *A* type is represented by an adequately large sample. The variability of the population is distinctly greater than that of the standard series Egyptian of 26th—30th Dynasty skulls, but rather less than that of a 17th century London series from a single cemetery. In general the more primitive existing races of man show smaller variability than advanced races, but it must be remembered that in the recent history of the world the former have been more restricted by inhabiting islands and more inaccessible regions and that they have had fewer opportunities of establishing contact with neighbouring races. The

* If Krause did not measure the parieto-squamous breadth, then his must have been less than the bi-parietal and greater than the maximum frontal and bi-asterionic breadths. I have only been able to find one measurement which would satisfy these conditions and which was fairly commonly used before 1897. Hermann Welcker (*Untersuchungen über Wachsthum und Bau des menschlichen Schädels. Erster Theil.* Leipzig, 1862, p. 24) defines a breadth taken between the points of intersection of a horizontal and a transverse circumference. Rudolf Virchow (Die altnordischen Schädel zu Kopenhagen. *Archiv für Anthropologie*, Bd. IV. 1870, p. 60) mentions a breadth taken between the *tubera parietalia*, but he uses the maximum bi-parietal breadth.

† See *Biometrika*, Vol. XVIII. 1926, pp. 415—416.

evidence is not inconsistent with the hypothesis that all races have potentially the same inherent faculty of variation, while the actual variability of each is determined by the conditions, whether ethnic or environmental, to which it has been subjected. The measurements of Australian skulls which have been published by 20 different anthropologists are in very satisfactory agreement except in one particular. An unsexed series described by Krause has means which are consistent with the others except for one dimension—the calvarial breadth. The present writer has been unable to escape from the conclusion that Professor Krause had measured that diameter in some most unusual way and not by the method which he professes to have used, but any more satisfactory explanation of the discordance would be greatly welcomed.

It is not possible to test the homogeneity of the native population of Tasmania by dividing into groups the sample of skulls of which measurements have been provided. The variability indicated is rather less than that of the Australian *A* type, but still distinctly greater than that of the Egyptian population. The Tasmanian and Australian skulls are quite widely differentiated and perfectly significant differences are found between the cephalic indices and several other calvarial and facial measurements. It cannot be said that the Tasmanian is nearer to the one Australian type than to the other. All three doubtless belong to the same family of prognathous Oceanic races. A comparison with the mean measurements of other types* shows that they have one or two extreme measurements, but the greater number of their characters fall well within the inter-racial distributions. The Australian *A* skull has a peculiarly small index 100 $L/S$ dependent on its low vault and retreating frontal bone, but it is not quite extreme and its height-length index (100 $H'/L$) is less remarkable. But for the same type the proportion of the bi-asterionic to the calvarial breadth is apparently greater than for any other modern race. The Tasmanian nasal index (100 $NB/NH'$) also seems to be the greatest recorded. Both Australians and Tasmanians have peculiarly small foraminal lengths (*fml*). The view advocated by some writers that the two races are distinguished from all other modern races on account of their ultra-primitive characters can receive no support whatever from a comparison of these measurements.

* The material for other races used in this comparison was that collected by the present writer in: Studies of Palaeolithic Man. II. *Annals of Eugenics*, Vol. II. 1927, pp. 318—380.

# MISCELLANEA.

## I.

## Further Contributions to the Theory of Small Samples.

(Vol. XVII. pp. 176 *et seq.*)

I regret to say that an error has crept into the above paper on p. 179. In my Lecture Notes from which this paper was printed the equation line 3 is correctly given as

$$F(\tfrac{1}{2}n, \tfrac{1}{2}n, \tfrac{1}{2}(n-1), \rho^2) = (1-\rho^2)^{-\frac{1}{2}(n+1)}\,\Gamma(-\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}(n-1), \rho^2),$$

and not equal to $(1-\rho^2)^{-\frac{1}{2}(n+1)}\,\Gamma(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}(n-1), \rho^2)$ as printed on p. 179.

Further the general term in (ix) is correctly given as it is below:

$$H(\rho^2) = \left(1 + \frac{\rho^2}{1!}\frac{1}{2n-2} + \frac{\rho^4}{2!}\frac{1}{(2n-2)(2n+2)} + \ldots + \frac{\rho^{2p}}{p!}\frac{1^2 . 3^2 . 5^2 \ldots (2p-3)^2}{(2n-2)(2n+2)\ldots(2n+4p-6)} + \ldots\right).$$

Very unfortunately the general term was overlooked in the next step in the work, and the

TABLE I (*Biometrika*, Vol. XVII. p. 182, revised).

*Values of the Correlation of Standard Deviations in Samples for Different Values of Variate Correlation in a Sampled Normal Population.*

*Values of* $\rho$.

| $n$ | ·0 | ·1 | ·2 | ·3 | ·4 | ·5 | ·6 | ·7 | ·8 | ·9 | 1·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | ·0000 | ·0088 | ·0352 | ·0796 | ·1421 | ·2239 | ·3261 | ·4501 | ·5988 | ·7772 | 1·0000 |
| 3 | ·0000 | ·0092 | ·0367 | ·0828 | ·1479 | ·2326 | ·3376 | ·4642 | ·6141 | ·7905$^-$ | 1·0000 |
| 4 | ·0000 | ·0094 | ·0375$^+$ | ·0846 | ·1510 | ·2371 | ·3435$^-$ | ·4712 | ·6216 | ·7965$^+$ | 1·0000 |
| 5 | ·0000 | ·0095$^-$ | ·0380 | ·0857 | ·1528 | ·2398 | ·3470 | ·4754 | ·6258 | ·7999 | 1·0000 |
| 6 | ·0000 | ·0096 | ·0383 | ·0864 | ·1540 | ·2415$^+$ | ·3493 | ·4783 | ·6285$^-$ | ·8019 | 1·0000 |
| 7 | ·0000 | ·0096 | ·0386 | ·0870 | ·1549 | ·2428 | ·3510 | ·4799 | ·6304 | ·8032 | 1·0000 |
| 8 | ·0000 | ·0097 | ·0388 | ·0873 | ·1556 | ·2438 | ·3522 | ·4813 | ·6317 | ·8042 | 1·0000 |
| 9 | ·0000 | ·0097 | ·0389 | ·0876 | ·1561 | ·2445$^-$ | ·3531 | ·4823 | ·6327 | ·8050$^-$ | 1·0000 |
| 10 | ·0000 | ·0097 | ·0390 | ·0879 | ·1565$^-$ | ·2451 | ·3538 | ·4832 | ·6335$^+$ | ·8055$^+$ | 1·0000 |
| 11 | ·0000 | ·0098 | ·0391 | ·0881 | ·1568 | ·2455$^+$ | ·3544 | ·4838 | ·6342 | ·8060 | 1·0000 |
| 12 | ·0000 | ·0098 | ·0392 | ·0882 | ·1571 | ·2459 | ·3549 | ·4844 | ·6347 | ·8064 | 1·0000 |
| 13 | ·0000 | ·0098 | ·0392 | ·0884 | ·1573 | ·2462 | ·3553 | ·4848 | ·6351 | ·8067 | 1·0000 |
| 14 | ·0000 | ·0098 | ·0393 | ·0885$^+$ | ·1575$^+$ | ·2465$^-$ | ·3557 | ·4852 | ·6355$^+$ | ·8069 | 1·0000 |
| 15 | ·0000 | ·0098 | ·0393 | ·0886 | ·1577 | ·2467 | ·3560 | ·4856 | ·6358 | ·8072 | 1·0000 |
| 16 | ·0000 | ·0098 | ·0394 | ·0887 | ·1578 | ·2469 | ·3562 | ·4859 | ·6361 | ·8074 | 1·0000 |
| 17 | ·0000 | ·0099 | ·0394 | ·0888 | ·1580 | ·2471 | ·3564 | ·4861 | ·6364 | ·8075$^+$ | 1·0000 |
| 18 | ·0000 | ·0099 | ·0395$^-$ | ·0888 | ·1581 | ·2473 | ·3566 | ·4863 | ·6366 | ·8077 | 1·0000 |
| 19 | ·0000 | ·0099 | ·0395$^-$ | ·0889 | ·1582 | ·2474 | ·3568 | ·4865$^+$ | ·6368 | ·8078 | 1·0000 |
| 20 | ·0000 | ·0099 | ·0395$^+$ | ·0890 | ·1583 | ·2476 | ·3570 | ·4867 | ·6369 | ·8079 | 1·0000 |
| 21 | ·0000 | ·0099 | ·0395$^+$ | ·0890 | ·1584 | ·2477 | ·3571 | ·4869 | ·6371 | ·8080 | 1·0000 |
| 22 | ·0000 | ·0099 | ·0396 | ·0891 | ·1584 | ·2478 | ·3573 | ·4870 | ·6372 | ·8081 | 1·0000 |
| 23 | ·0000 | ·0099 | ·0396 | ·0891 | ·1585$^+$ | ·2479 | ·3574 | ·4871 | ·6374 | ·8082 | 1·0000 |
| 24 | ·0000 | ·0099 | ·0396 | ·0891 | ·1586 | ·2480 | ·3575$^-$ | ·4872 | ·6375$^-$ | ·8083 | 1·0000 |
| 25 | ·0000 | ·0099 | ·0396 | ·0892 | ·1586 | ·2481 | ·3576 | ·4873 | ·6376 | ·8084 | 1·0000 |
| 50 | ·0000 | ·0100 | ·0398 | ·0896 | ·1593 | ·2490 | ·3588 | ·4887 | ·6388 | ·8092 | 1·0000 |
| 100 | ·0000 | ·0100 | ·0399 | ·0898 | ·1597 | ·2495$^+$ | ·3594 | ·4894 | ·6394 | ·8096 | 1·0000 |
| 400 | ·0000 | ·0100 | ·0400 | ·0900 | ·1599 | ·2499 | ·3599 | ·4899 | ·6399 | ·8099 | 1·0000 |
| $\rho^2$ | ·0000 | ·0100 | ·0400 | ·0900 | ·1600 | ·2500 | ·3600 | ·4900 | ·6400 | ·8100 | 1·0000 |

first three terms only retained, from which a new general term appears to have been written down and given to the computer of Table I on p. 182 of the paper. The last column of that Table had previously troubled me, but I had not perceived the source of the error until Mr J. Wishart kindly drew my attention to the source of the trouble in the general term of equation (ix). I am very grateful to Miss M. Moul for a recomputation of the Table, which has been considerably modified in the right-hand upper region.

Mr Wishart among other elegant results of a memoir to be published in the next number of *Biometrika* has shown that the correlation of the variances, $r_{\sigma_x^2 \sigma_y^2}$, where selection is made from an infinite normal population, is given by

$$r_{\sigma_x^2 \sigma_y^2} = \rho^2_{xy},$$

which had hitherto only been obtained by approximate methods. I had suggested to him that this could be confirmed by my equation (v) of p. 177 of the paper cited, which gives the correlation surface of $\sigma_1$, $\sigma_2$ (and therefore of course of $\sigma_1^2$, $\sigma_2^2$) in samples of any size. This he found followed easily. I have since succeeded in showing from the same equation (v) that the regression lines of the variances are straight.

In fact one finds, if $\bar{\sigma}_2^2$ be the probable value of $\sigma_2^2$ for a given $\sigma_1^2$, that:

$$\bar{\sigma}_2^2 = \left(1 - \frac{1}{n}\right) \Sigma_2^2 (1 - \rho^2) + \frac{\Sigma_2^2}{\Sigma_1^2} \rho^2 \sigma_1^2,$$

where $\Sigma_1$, $\Sigma_2$ are the standard deviations and $\rho$ the correlation in the population sampled.

If on the other hand we take the correlation of $\sigma_1$ and $\sigma_2$, or $r_{\sigma_1 \sigma_2}$, we find that the value of $r_{\sigma_1 \sigma_2}$ changes with the size of the sample $n$ as well as with the correlation $\rho$ of the variates in the sampled population, as the Table indicates. Further, the regression is not linear, but of rather a complicated nature. It might therefore be thought advisable to use the variances rather than the standard deviations. Unfortunately it is the latter rather than the former which more frequently occur in the problems to which such results must be applied. K. P.

## II.

In discussing in the July issue of this Journal* the work of A. J. Lotka and Vito Volterra regarding the interdependence of species living together, I stated that while the latter obtained the fundamental differential equation, (viii) of my article, as a reasonable form of relation to be expected, based on the hypothesis of encounters, the former reached it only as an approximation derived from a Taylor expansion of the more general theoretical equation. I have to thank Dr Lotka for since pointing out to me that I had overlooked the fact that he has himself considered this aspect of the problem in a later section of his book (Chapter XXV, p. 359), where he shows how the interpretation of encounters or of collisions and capture may be given to the term $KN_1N_2$, reached in the earlier part of the book without a precise analysis of its physical significance. In this later section Dr Lotka also discusses the effect of velocity of motion upon frequency of encounter. E. S. P.

*Erratum.*

Mr J. Wishart wishes to draw attention to a slip in his memoir in Vol. XIX. p. 27, two lines below equation (26), where the sign of the radical has been omitted in the value of $u_s$, which should read

$$u_s = \frac{mx}{\sqrt{l+s+1}}.$$

This correct value is applied in the examples on p. 28.

Subscribers to Vol. XIX are requested to make this change in the text.

* *Biometrika*, Vol. XIX. pp. 216—222.

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